Communities, Co-ops, and Clubs: Social Capital and Incentives in Large Collective Organizations

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We study a continuous-time organization design problem. Each member's output is an imperfect signal of his underlying effort, and each member's utility from remaining in the organization is endogenous to other members' efforts. Monetary transfers are assumed infeasible. Incentives can be provided only through two channels: expulsion following poor performance and respite following good performance. We derive the steady state distribution of members' continuation utilities for arbitrary values of the initial and maximum continuation utilities and then optimize these values according to organizational objectives. An optimally designed organization can be implemented by associating continuation utilities with a performance-tracking reputation system.
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> If the members of a large group rationally seek to maximize their personal welfare, they will *not* act to advance their common or group objectives unless there is coercion to force them to do so, or unless some separate incentive, distinct from the achievement of the common or group interest, is offered to the members of the group individually on the condition that they help bear the costs or burdens involved in the achievement of the group objectives.
> —Mancur Olson, The Logic of Collective Action: Public Goods and the Theory of Groups (1971)

Organizations where members share access to a collectively produced common good are ubiquitous: e.g., communities, co-ops, clubs, and teams. Such collectives are often egalitarian in the sense that output is shared more or less equally among members. This can occur either for technological reasons (when a team wins, all its members enjoy the victory), or for ideological reasons (the output of communal

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farms such as the Israeli Kibbutzim or North-American Hutterite settlements customarily accrues equally to all adult members (Van den Berghe and Peter, 1988; Abramitzky, 2008)). It is, therefore, often either infeasible or undesirable to provide incentives to members of a collective through explicit monetary channels. Rather, the organization may track the performance of individual members and may use a variety of non-pecuniary incentives such as rewarding good performance with respite or prestige and punishing poor performance with peer pressure or — in extreme cases — expulsion (Freeman, Kruse and Blasi, 2008).

In this paper we analyze a continuous-time organization design problem where a social planner must manage incentives for a continuum of agents — the members — who may make costly contributions to each other's payoffs in the form of contributing to the commonly consumed good. Such effort is not observed directly, but rather with noise, and thus each agent's performance is stochastic. To provide incentives, the designer maintains a (possibly informal) reputation system tied to agents' performance, whereby agents who hit the bottom of the reputation scale are expelled from the organization and agents who hit the top *must* be allowed a respite, that is, to shirk for a nonnegligible amount of time. At the top of the scale, an agent's situation can improve no further, and thus it is impossible to incentivize effort. Between these extreme ends of the scale, agents are expected to exert full effort. Because expulsions and respites are both socially inefficient, each agent's reputation is linked to his stochastic output with the minimal sensitivity consistent with incentive compatibility.

Our main insight is that collective organizations feature a particular dynamic feedback effect, and thus their optimal design often involves a novel trade-off not present in standard principal-agent settings. In collective organizations, the value of being a member is endogenously determined as a function of the efforts of all other members. Absent monetary transfers, such an organization can mimic the effect of bonus payments for good performance by allowing agents to shirk occasionally; however, shirking by some agents reduces the value to *all* agents of being in the organization. In turn, this reduces members' fear of being expelled for bad performance, further undermining incentives.

We model organizations as having an exogenous inflow of new members, which is meant to capture the fact that individuals hear about the organization or develop an interest in it over time. The instantaneous effort of each member of the organization is confounded with normally distributed noise, and thus a given agent's performance and continuation payoff are Brownian diffusions. Given the law of motion of agents' continuation values under the reward and punishment scheme described above, we employ methods in stochastic processes to derive the steady state in which the density of organization members at each level of reputation (or continuation utility) remains constant over time. The characteristics of the steady state distribution are governed by two parameters: the highest attainable level of continuation utility, w^*, and the level at which new organization members are inserted, w^0. We then use this steady state to frame the designer's problem directly, which — as discussed further below — may involve a variety of different objective functions.

The steady state distribution exhibits several critical features. First, by definition, the continuation value $w = 0$ is a *resetting boundary* in the sense that the flow of agents being expelled from the organization at that point must equal the flow of new agents joining at the higher level w^0. Second, w^* is a *sticky boundary* or *slow-reflecting barrier* in the sense that an agent reaching this level accumulates a positive (finite) expected measure of time there. Moreover, an agent exerts low effort — *shirks* — if and only if his continuation utility is exactly w^*, and otherwise he provides high effort — *works.* Together these observations imply that the steady state distribution possesses a positive mass of agents at the very top of the scale who are rewarded with a respite from working. Importantly, because we are studying an organization where efforts have externalities, these reward periods affect all members and their incentives by reducing their collective flow benefits. For the organization to be sustainable, agents must earn a positive continuation payoff from membership.[1] This places a natural limit on the fraction of members who feasibly can be permitted to free-ride at any given instant.

Our primary goal is to investigate a social planner's organization design problem; namely, we allow her to choose w^* and w^0 subject to feasibility constraints. To build intuition, and as an intermediate step in solving the problem, it is useful to consider a relaxed problem whereby the planner promises agents a fixed, positive flow payoff from membership in the organization. This results in a bounded set of design parameters, w^0 and w^*, for which the organization generates enough output to cover the promised flow payments. We show that when the principal increases w^0, the starting value to new agents, both the mass of working agents and the mass of shirking agents increase, and in fact, the latter effect dominates with respect to the fraction of agents shirking; the planner may, therefore, be forced to limit the starting utility assigned to new members in order for generated output to cover the promised flow to all members. Increasing the maximum continuation value w^* means promising later, but more frequent, periods of respite; this change in the organization also increases both the mass of working agents and the mass of shirking agents. The latter effect eventually dominates, and the average output of the organization falls. Hence, the principal must also limit the maximum continuation value to agents in the organization, which in turn limits their expected tenure and ultimately the overall size of the collective.

We then consider two possible objective functions for the planner. For instance, a planner who generates revenue primarily through advertising (e.g., a social network) would want to maximize the size of the organization. In this case, the objective is increasing in both w^* and w^0: increasing these values delays the time at which agents are expelled, which leads to a larger organization in the steady

[1] Our participation constraint assumes that agents are free to exit the organization at will. This was not the case, for example, in the collective farms of China in the 1960s and 70s, where peasants, who would have preferred to migrate to cities, were compelled to live and work on their assigned rural plantations (Zhao, 1999).

state. The feasibility constraint then forces the planner to trade-off higher w^0 with higher w^*. On the other hand, a social planner may — as in a partnership or co-op setting — want to maximize the per-capita output of members, which would involve maximizing the fraction of agents working.[2] We show that this objective leads to a vanishingly small organization; members are promised an arbitrarily small continuation value with virtually no hope of earning an opportunity to free-ride before being dismissed. Whatever her objective, we note that by identifying continuation utilities with reputation scores, the planner's problem can be thought of as the design of an optimal reputation system.

We discuss applications of our model and related literature in the next section. In Section II we introduce the formal model along with a broad overview of the steps involved in the analysis. In Section III we solve for the steady state distribution of continuation utilities (i.e., reputations) of the organization members for arbitrary values of the policy instruments. Section IV characterizes the feasible set of choice variables. In Section V we discuss the principal's organization design problem for the two possible objectives mentioned above. We summarize our findings and outline some directions for future work in Section VI.

I. Applications and Related Literature

A. *Applications*

Our setting has three key features that distinguish it from standard dynamic contracting models: (i) agents decide whether to exert effort, which has positive externalities, (ii) the organization designer has limited incentive instruments, namely respite and expulsion, to incentivize agents to exert effort, and (iii) the designer chooses how to use these incentive instruments, taking into account how agents will behave in response. Numerous organizations exhibit such traits to some degree.

Feature (i) can arise in organizations in two ways. First, in some organizations, agents' efforts contribute to a common good which is divided evenly among all members. As noted above, in farming collectives such as the kibbutzim and Hutterite settlements, output — or the proceeds from selling output — is generally shared equally among all adults (Van den Berghe and Peter, 1988; Abramitzky, 2008). Likewise, when members of a club exert effort to promote an event or engage in fundraising activities, all members of the club benefit. Second, in some organizations, such as ride-sharing or room-rental platforms, agents are paired with one another in short-term matches, and each agent's effort benefits solely her current partner. For example, in the case of a ride-sharing platform, positive

[2]The literature on the neoclassical theory of labor managed firms uses the term *Illyrian firm* to refer to those which maximize *dividends* or net revenue per worker (Ward, 1958; Law, 1977). However, see our discussion of the objectives actually pursued by worker co-ops at the end of the following subsection.

externalities within a match are associated with the efforts of both drivers and passengers to be prompt and courteous.

Feature (ii) is motivated by the observation that transfers are notably absent as incentive instruments in many organizations. The kibbutzim and Hutterite settlements, having been founded on egalitarian ideals, are opposed to transfers by their very nature. Sharing economy platforms typically do not use bonus payments based on ratings to incentivize their users.[3]

Feature (iii) forms the essence of organizational design. For instance, Uber maintains a five-star rating system for both drivers and passengers, with 5.0 being the maximum possible rating. In September 2018, Uber introduced a policy in Australia and New Zealand, where the average user's rating is 4.5, of deactivating users whose ratings fall below a minimum level (Cherney, 2018). In an example where the community is interpreted at a national scale, China has introduced a social credit rating system for all its citizens which takes into account both financial behavior (such as credit-card payments) and social behavior (such as volunteer activity and adherence to family-planning limits); this system integrates with already existing blacklisting systems used to restrict activities such as loans and travel (Chin and Wong, 2016). In both of these examples the function of the rating system is to track performance and incentivize good behavior as defined by the designer.

Perhaps the best overall application of our model is to a worker cooperative or labor-managed firm (LMF).[4] These entities are generally founded on egalitarian principles and therefore exhibit a high degree of equality in pay among members regardless of seniority or skill differentials (see Chapter 6 in Bonin and Putterman (2013) for numerous examples and insightful discussion). Thus, a typical LMF produces output that is shared more or less equally by its members and does not provide incentives through explicit monetary channels. Monitoring in modern LMFs is often performed by peers where: "most workers say that they can detect fellow employees who shirk … and many report that they would speak to the shirker or report the behavior to a supervisor" (Freeman, Kruse and Blasi, 2008, p. 1). While termination appears to be more rare in LMFs than in conventional firms (Alves, Burdín and Dean, 2016), the ultimate threat of dismissal still plays an important incentive role in a variety of co-op settings (Albanese, Navarra and Tortia, 2017). This is especially true for junior or candidate members who often face a probationary period before becoming vested members of the enterprise.

[3] To be sure, ride-sharing platforms such as Lyft, Gett, Juno, Uber and Via, do allow passengers to tip drivers through their apps, but a recent survey of over 2600 active Uber drivers revealed that "…tip income was negligible in the majority of cases." (Wong, 2018)

[4] The most authoritative empirical research on worker cooperatives in the US is Craig and Pencavel (1992) who investigated the behavior of the largest and most durable LMFs in US manufacturing, the plywood firms in the Pacific Northwest. Three other examples of historically prosperous co-ops, worker-owned scavenger companies, taxi cooperatives, and professional partnerships such as legal or accounting firms, are discussed by Russell (1985). Non-profit hospitals were modeled as physician cooperatives by Pauly and Redisch (1973), and university academic departments were modeled as faculty co-ops by James and Neuberger (1981).

However, "shareholders can also be fired for repeated malfeasance" (Craig and Pencavel, 1992, p. 1084). For instance, Bylaw 21 of the shareholder's manual issued by the Fort Vancouver Plywood Company, Inc. provides:

> The board of directors at any regular or special meeting shall have the power by a majority vote to remove from working status any shareholder-worker whom they shall find to be physically or mentally unfit for such work, or who refuses to do his work as outlined by the management.[5]

While it is difficult to document explicit respite policies for good performance as such, Craig and Pencavel (1992, p. 1085) report, "Job assignments are varied and sometimes rotated, although, if a particularly attractive position opens up, its allocation is determined by seniority or previous work performance." Thus, the structure and internal governance of LMFs appear to square remarkably well with the three distinguishing features of our model: team production, non-pecuniary incentive instruments, and implementation of incentives via imperfect dynamic monitoring.

The question of *optimal* organization design presupposes an objective function for the designer, and indeed, there has been considerable theoretical debate about the objectives of LMFs ever since the seminal publication by Ward (1958).[6] To address the question empirically, Burdín and Dean (2012) use panel data from 31 Uruguayan industries to estimate the relative weight worker-managed firms place on total employment (firm size) versus dividends (net revenue per worker). Interestingly, the LMFs in their data appear to place relatively high weight on the size of the enterprise, employing systematically more workers than would comparable profit-maximizing firms. In a sense, this observation also squares with our results. Specifically, we show in Proposition 8 that an organization that endeavors to maximize per capita output without placing any weight on employment must be arbitrarily small.

[5]See http://courts.mrsc.org/appellate/024wnapp/024WnApp0120.htm. There is evidence that this provision was indeed sometimes used to dismiss vested members of the cooperative who were detected shirking. In one such case a dismissed worker's foreman testified in court:

> Well, he was placed on the 8-foot green chain during the three days. I observed his work habits and they were very poor. And I received numerous complaints from the other green chain workers. And so I told Mr. McIntyre two or three times he was going to have to improve, work harder, just show more initiative. The following two days I didn't see any initiative at all. The other four guys had to do all the work. And I had gone to the superintendent on a couple of times and asked him to come down and watch and see what kind of problems I had. So he did come down and observed his work habits a couple times before we decided on the pink slip.

[6]The objective functions we consider, namely organizational size and per capita output, are analogous to the two sides of this debate.

B. Literature

Besides the literature discussed in the previous subsection, this paper also contributes to several additional lines of research. The first, on the economics of clubs and other collective organizations more generally, begins with the classic works of Buchanan (1965) and Olson (1971), along with Helsley and Strange (1991), Scotchmer (1985), and Oakland (1972) among others, dealing with optimal group membership, size, and fee structure.

A second literature to which we contribute, on the use of ratings and reputation for incentive compatibility, begins with Holmström (1999) and includes — among others — a recent working paper by Hörner and Lambert (2016). This literature investigates the use of reputation as a means for eliciting the rated agent's cooperation in a setting where the principal has imperfect control over compensation. In order to maintain a high reputation — and thereby a high continuation payoff — agents are required to produce a stream of signals reflective of high effort.[7] Using different terminology, Olszewski and Safronov (2018) study the use of *chips* for incentives in a favor-exchange game between two players.

Another branch of related research studies continuous-time optimal contracting in the context of corporate finance. The pioneering article in this literature is DeMarzo and Sannikov (2006) (hereafter DS), which was followed by a number of related works including Sannikov (2008), Zhu (2013), and Grochulski and Zhang (2016).[8] Each of these papers investigates variations of the DS baseline model which we, too, adapt to our setting. Specifically each of them considers a single agent who may take an action either to produce output or to benefit himself, and solves for the optimal path of continuation values in order to maximize output. DS implement their optimal contract by way of basic financial instruments, while the others abstract from implementation considerations.

We also contribute to a recent burgeoning literature on dynamic incentives in the absence of monetary transfers. For instance, Li, Matouschek and Powell (2017) study the dynamic allocation of power between a principal, who has formal authority in the organization, and an agent, who has private information regarding the current set of available projects. Under the optimal relational contract, the agent recommends a project to be completed each period. The more frequently he recommends the principal's favorite project rather than his own, the more continuation utility the agent accumulates. High values of continuation utility are associated with more power in the organization in the sense that the principal is obliged to accept the agent's recommendations with greater frequency. The authors show that this process ultimately converges to one of two (inefficient) absorbing states: a maximum level of continuation utility for the agent that is implemented by always accepting his recommendations or a minimum level of

[7] An alternative use of ratings is studied by Bonatti and Cisternas (2018).

[8] See also the literature on folk theorems in continuous time with imperfect monitoring; e.g., Sannikov (2007), Peski and Wiseman (2015), and Bernard and Frei (2016).

continuation utility that is implemented by never listening to him. Another paper in this vein is Lipnowski and Ramos (2018) who explore delegated authority in an infinite-horizon game with imperfect monitoring. In this setting a principal wishes to fund good projects and not fund bad ones. An agent is privately informed about the state of each project, but prefers the principal to fund them all. In the initial phase of the relationship the agent is granted considerable authority to initiate projects and he accordingly directs the principal to fund all good ones as well as a significant fraction of those that are bad. As time progresses, the agent's goodwill eventually runs dry — the equilibrium of the game enters a phase where the principal rarely delegates authority to the agent, and when she does, the agent recommends only good projects. A third recent paper in this line is Guo and Hörner (2018) which investigates the limits to efficient dynamic allocation in the presence of private persistent information. In this model, a social planner with full power of commitment wishes to supply a perishable good to an agent in those periods when his valuation exceeds the constant provision cost. The agent's valuation is always positive but is not observed or learned, even imperfectly, by the planner. As in Li, Matouschek and Powell (2017), the optimal incentive compatible mechanism eventually converges to one of two antipodal (inefficient) situations: the agent either receives the good in every period *ad infinitum* or he never receives it again.

While we too consider dynamic incentives in the absence of monetary transfers, our focus differs from the three papers just outlined in several key respects. First, we consider a continuum of agents who interact with each other rather than focusing on providing incentives for a single agent in isolation. Second, the designer in our setting is interested in the optimal structure of the collective organization as represented by the steady-state equilibrium distribution of continuation utilities of its members. Finally, we study a hidden action model in continuous time rather than a hidden information model in discrete time.

Of course, our paper also contributes to the organizational economics literature. Specifically, because each agent in our model receives a gross expected payoff related to the effort provided by other agents, our setting resembles a dynamic *partnership* in which output is divided equally among the members of the organization as in Farrell and Scotchmer (1988) and Levin and Tadelis (2005), models of favor-trading as in Hauser and Hopenhayn (2010), and of dynamic partnership rematching as in McAdams (2011). This partnership aspect and the methods we use to derive and analyze the steady state are the key features that distinguish our model from other recent work on dynamic relational contracts such as Andrews and Barron (2016).

Finally, Acemoglu and Wolitzky (2018) model a society as a population of agents who can exert effort with positive externalities and who may be punished for exerting low effort; some fraction of the population are elite types who are less vulnerable to this punishment. The authors show that elites may nonetheless prefer to receive equal punishments — so-called equality under the law — as this increases their joint effort which then increases the effort of non-elites. In contrast to their model, ours features imperfect monitoring, which gives rise to an equilibrium distribution of continuation payoffs and effort dynamics that underlie the organizational design problem. Acemoglu and Wolitzky (2018) also assume the existence of an explicit punishment technology, while incentives in our setting are generated either by letting agents enjoy the public good without exerting effort (respite) or by cutting off their access to the public good entirely (expulsion).

II. Setup

Time is continuous over an infinite horizon. At each moment there is a positive measure of massless agents present in an organization. Agents, the members, are indexed by a continuous variable i. Each agent is risk neutral and discounts the future at rate r. A member who *exits* the organization receives a payoff of 0 from that point forward, and thus the rule that specifies when to remove an agent from the organization will be a key component of organization design. A flow of new members $\psi > 0$ join the organization at each instant; this will generate turnover of members while allowing us to use steady state methods.

While remaining in the organization, each agent receives a flow utility u which will, in equilibrium, be generated by the collective actions of all members. For now, we can assume u is a constant, exogenously determined flow payoff. At each instant, each agent i chooses an effort level $e^i \in \{H, L\}$, where we refer to choosing H as *working* and L, *shirking*; thus each agent i chooses a stream of effort levels, which is a stochastic process $(e^i_t)_{t\geq 0}$. The flow cost of effort is $c(e^i_t)$, defined by $c(H) = c > 0$ and $c(L) = 0$. Thus, high effort has a flow cost $c > 0$ and low effort has no cost. An agent's effort generates an output stream of contributions to the common good given by a Brownian diffusion

$$dX^i_t = (\mu_{e^i_t})dt + dB^i_t, \tag{1}$$

where we assume that $\mu_H - c > 0 > \mu_L$, so that high effort is efficient and low effort is not.

The diffusion process X^i admits multiple interpretations. First, in some instances, each agent's output may be clearly delineated from others'; for example, if each agent in a farming collective is responsible for an individual plot of land, each agent's output is clearly defined and can be monitored in isolation. Second, even if output is not clearly delineated — agents may be working in groups and their output jointly determined — the principal might have means of monitoring individual agents' efforts. Indeed, although we follow the convention of calling X^i the *output* stream of agent i, it can also be interpreted as a stream of signals about agent i's unobserved effort. Third, agents might observe (signals of) one another's efforts and report these to the principal. For example, in ride-sharing services, agents are randomly matched and observe noisy signals of their match's effort. With a large number of agents, repeated interactions are rare, and agents may be willing to truthfully report their signals to the principal.

Since agents with the same continuation utilities are essentially identical, we suppress the index i whenever doing so does not create confusion. Below we also speak of *the agent* with the understanding that we are focusing on a single arbitrary member of the organization. A *contract* in this context specifies: (i) a fixed flow utility u, (ii) a removal time τ and (iii) a recommended effort process e. We consider permanent expulsion at τ. Also, an agent is free to leave the organization at any point, but may not rejoin.

In order to implement a contract, the principal assigns each agent a score or *reputation* process, S. Each incoming agent begins with some initial reputation level, and his reputation evolves thereafter according to his output stream. As each agent is motivated solely by the evolution of his continuation payoff, the designer may simply set the reputation process S for each agent equal to his continuation payoff process W. In particular, (i) an agent is removed from the organization when his reputation reaches $w = 0$, (ii) new agents are granted a continuation payoff of $w^0 > 0$, and (iii) since flow payoffs are bounded above and agents discount the future, there is some maximum attainable reputation level w^*. The distribution of agents at each instant thus is characterized by a population distribution over continuation payoffs. We ultimately will allow the principal to directly choose w^0 and w^* as a part of the organization design problem.

From standard results in continuous-time contracting,[9] the following are known:

- While the agent remains in the organization (i.e., $t \leq \tau$), there exists a process β_t representing the sensitivity of the agent's continuation value to output:

$$dW_t = rW_t dt - (u - c(e_t))dt + \beta_t(dX_t - \mu_{e_t} dt).$$

- The contract is incentive compatible if and only if for all $t \leq \tau$ and $W_t \geq 0$, $e_t = H$ implies $\beta_t \geq \lambda$ and $e_t = L$ implies $\beta_t \leq \lambda$, where $\lambda := \frac{c}{\mu_H - \mu_L}$.

So long as the designer's objective is increasing in effort,[10] removing an agent is inefficient (i.e., on path, agents are removed due to bad luck, not because they were shirking at the moment). Therefore, the principal wishes to minimize volatility and thus minimize the sensitivity β_t subject to incentive compatibility for the recommended effort level; it is optimal to set either $\beta_t = \lambda$ to induce working or $\beta_t = 0$ to induce shirking. Hence, when the agent works, his continuation value evolves as

$$\begin{aligned} dW_t &= rW_t dt - (u - c)dt + \lambda(dX_t - \mu_H dt) \\ &= (rW_t - (u - c))dt + \lambda dB_t. \end{aligned} \tag{2}$$

[9]See DeMarzo and Sannikov (2006) Lemmas 2 and 3 or Zhu (2013) Lemmas 3.1 and 3.2.

[10]The designer's objective may be increasing in effort either directly, if effort enters into the objective function, or indirectly, if effort (of other agents) enters into agents' value of being in the organization.

When the agent shirks, his continuation value evolves deterministically as

$$dW_t = (rW_t - u)dt. \tag{3}$$

In particular, the agent shirks whenever his reputation reaches its maximum level $w^* < u/r$, where the drift in (3) is necessarily downward.[11] For later convenience, we define $\rho(w) := rw - u$ so that $\rho(w^*)$ is the (downward) drift at w^*.

To summarize, the contract terms can be expressed in terms of (u, w^0, w^*), where an individual agent's continuation value process W starts at w^0, it evolves according to (2) when $W_t \in (0, w^*)$ and (3) when $W_t = w^*$, and the agent is removed when $W_t = 0$. Formally, the process W belongs to a class of diffusions known as *Sticky Brownian Motion*.[12]

A. *Overview of Analysis*

Now that the model has been presented, it is helpful to outline the four steps we use to solve the organization design problem.

1) *Determine agents' law of motion*: fixing contract terms (u, w^0, w^*), we characterize the evolution of the state variable — i.e., continuation utility — for each agent as a stochastic process. This step has been performed above.

2) *Find steady state distribution*: we next determine a stationary distribution for continuation utilities under the contract terms (u, w^0, w^*), using the law of motion above and taking into account the inflow and outflow of agents.

3) *Determine feasibility*: since agents in a collective organization exert externalities on one another, the flow payoffs u they earn must be consistent with the steady state distribution of agents and their behavior. This requirement determines a feasible set of organization parameters (u, w^0, w^*) over which the designer can optimize.

4) *Optimize organization parameters*: for a given objective function for the organization designer, we determine the optimal values of the design parameters within the feasible set derived in the third step.

III. The Steady State

A steady state corresponds to a stationary distribution of agents over all possible levels of continuation payoff, as poorly performing agents drop out of the

[11] Recall that u/r is the value an agent would get by shirking yet remaining in the organization forever, and is therefore an upper bound on continuation payoffs. Since we will pose the principal's problem directly in terms of a steady state distribution of agents, we cannot have $w^* = u/r$; otherwise, the drift in (3) would vanish for $W_t = w^*$, making w^* an absorbing state, leading to an organization that grows over time without bound.

[12] See Harrison and Lemoine (1981) and Zhu (2013).

organization, new agents arrive, and as the continuation value of all agents within the system evolve in response to their Brownian output streams. Essentially, a stationary distribution is a distribution of agents' continuation values in $[0, w^*]$ in the organization which is constant over time; any movement of agents away from a particular continuation value (up or down) is exactly offset by movement toward that continuation value by other agents. As we will be interested in the total mass of agents in the organization as well as their distribution, we do not require that the stationary distribution integrate to one. Mathematically, the steady state in our setting is equivalent to a rescaling of the stationary distribution of a process W defined on the interval $[0, w^*]$ as follows:

- When $W_t = 0$, it immediately resets to w^0.
- For $W_t \in (0, w^*)$, it evolves as (2).
- For $W_t = w^*$, it evolves as (3).

In other words, the process undergoes resetting (as exiting agents are replaced with new agents) and slow reflection (as agents at the top of the distribution are permitted to shirk).[13] The stationary distribution is scaled so that the flow rate of mass at 0 is ψ, the inflow of agents into the organization.

Proposition 1 fully characterizes the distribution of continuation utilities in a steady state for a given specification of the promised flow payoff u, the highest achievable continuation utility w^*, and the level at which new agents are admitted to the organization w^0. The proof is given in the appendix.

Define $\gamma(w) := \frac{rw-(u-c)}{\lambda\sqrt{r}}$ and $\text{erf}\,\{x\} := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, the Gauss error function.

Proposition 1. *In a steady state, the distribution of agents consists of two densities f_-, f_+ and an atom $\nu\{w^*\}$ given by*

- $f_-(w) = e^{\gamma(w)^2}\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\,\{\gamma(w)\}\right]$ *for* $w \in [0, w^0]$;
- $f_+(w) = e^{\gamma(w)^2}\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right]$ *for* $w \in [w^0, w^*)$;
- $\nu\{w^*\} = \frac{\lambda^2 f_+(w^*)}{2(u-rw^*)}$.

In words, the distribution consists of a density function composed of two segments (with a kink where they meet at w^0) as well as a mass point at the top of the support, w^*; Figure 1 illustrates.

We make several observations about the stationary distribution. In discrete time, the continuation value of each agent follows a random walk. At generic states $w \notin \{0, w^0, w^*\}$, agents "leaving" the continuation value w are immediately replaced by agents moving up from $w - dw$ or down from $w + dw$. In continuous

[13] A combination of resetting and (partial) reflection arises Kolb (2019), where the underlying process is a seller's reputation.

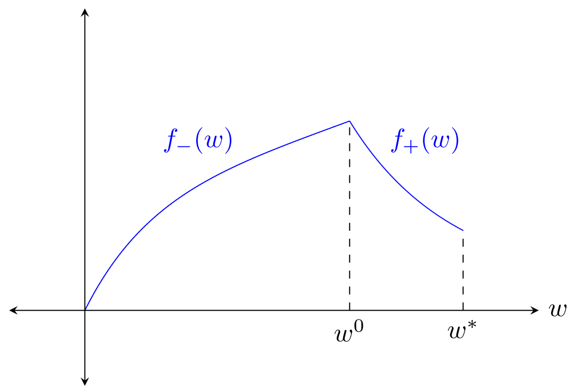

Figure 1. : Steady state distribution of agents (omitting the atom at w^*).

time, this is stated by the Kolmogorov forward equation (A1), which implies the functional form for the densities $f_-(w)$ and $f_+(w)$ up to two constants. The difference in slopes at the kink at w^0 is due to the exogenous inflow of new agents and must match the outflow of agents at $w = 0$.

The third item of Proposition 1 is the most important economically. First, note that there is a strictly positive measure of agents at w^* who are shirking; since the process W is a sticky Brownian motion, there is a positive measure of times t at which $W_t = w^*$, and thus the stationary distribution involves an atom of mass at exactly w^*. This means that there is a nontrivial fraction of agents in the organization who are shirking at any time. Second, the size of this atom $\nu\{w^*\}$ is proportional to the "stickiness" $u - rw^*$ of the process at w^*. The greater the maximum continuation value w^*, the more "frequent" the shirking reward must be delivered to sustain it (the continuation value process is stickier at w^*), and thus the larger the measure of agents who are shirking in a steady state.

Given the steady state for exogenous organization design parameter values (u, w^0, w^*), the question now is what values the principal can and should set.

IV. Feasibility of Organizations

Prior to this point it has sufficed to interpret u as an exogenous flow payoff that the principal can promise the agents. In this section, we endogenize this flow payoff by imposing the *feasibility* constraint that the principal can only promise what flow value the organization itself produces. We obtain, in closed form, a sufficient condition and a necessary condition for the existence of a feasible organization. Moreover, we characterize the way in which feasibility depends on the organization design parameters, and provide further qualitative results about the set of feasible organizations.

To capture in a tractable model the feature that working agents impose positive externalities on the platform and shirking agents impose negative externalities, we specify that the flow payoff each agent receives is the average output of all the other agents. We offer two interpretations for such payoffs. One interpretation is that all agents contribute their efforts high or low to a common good which gets divided evenly among all agents in the organization, and thus agents obtain a flow value equal to the average flow output of its members. A second interpretation is that agents in the organization are randomly and instantaneously matched with one another, and agents obtain a flow value equal to the output of their partner. In expectation, each agent receives the weighted average output of other agents.

Since the average output of all agents is a function of the mass of working agents and mass of shirking agents, and those masses are in turn functions of the promised flow payoff u, the latter is now a fixed point.

We now formally analyze feasibility. Define the measure of agents active (i.e., working) in a steady state by

$$\alpha(u, w^0, w^*) := \int_0^{w^0} f_-(w)\, dw + \int_{w^0}^{w^*} f_+(w)\, dw,$$

and (with some abuse of notation) define the measure of non-active (i.e., shirking) agents at the top of the distribution by

$$\nu(u, w^0, w^*) := \nu\{w^*\}.$$

Denote the fraction of agents working by

$$Q(u, w^0, w^*) := \frac{\alpha(u, w^0, w^*)}{\alpha(u, w^0, w^*) + \nu(u, w^0, w^*)}. \tag{4}$$

The endogenous flow utility produced when fraction $Q = Q(u, w^0, w^*)$ of agents are working is thus the average output,

$$U(Q) := Q\mu_H + (1 - Q)\mu_L. \tag{5}$$

where working agents contribute μ_H and shirking agents contribute μ_L.

In order to define feasibility, we state two conditions:

$$0 < w^0 \leq w^* < u/r \tag{6}$$

$$U(Q(u, w^0, w^*)) = u. \tag{7}$$

Definition 1. *An organization (u, w^0, w^*) is feasible if (6) and (7) are satisfied.*

The key part of Definition 1 is the fixed point condition $u = U(Q(u, w^0, w^*))$. This condition says that the flow payoff that each agent receives, which the principal has exogenously promised, must equal the flow payoff generated by the organization. Condition (6) says that u must be positive, otherwise agents would be better off leaving the organization (individual rationality) than working at all. Moreover, $w^* < \frac{u}{r}$ because the highest level of continuation utility can at most equal the perpetuity value of shirking forever (and as noted earlier, if $w^* = u/r$, w^* becomes an absorbing state). We denote by $\mathcal{S}$ the set of feasible organizations.

A. Existence and Nonexistence

Since an organization design (u, w^0, w^*) is feasible only if u is generated by the agents in the organization, a feasible organization need not exist. The next proposition ensures that when agents are sufficiently patient, when the cost of effort is sufficiently low, or when effort produces sufficiently high output, a feasible organization exists; that is, there exists a triple (u, w^0, w^*) that satisfies (6) and (7).[14] Note that in the extreme case, if agents were perfectly patient, the principal could sustain an arbitrarily large organization with an arbitrarily large fraction of agents working; in that case, the prospect of expulsion alone would incentivize agents to work.

Define

$$r^* := \frac{2(\mu_H - \mu_L)^2}{c}\left(c + \mu_H - 2\mu_L - 2\sqrt{(c-\mu_L)(\mu_H - \mu_L)}\right). \tag{8}$$

Proposition 2. *If $r \in (0, r^*)$, then the feasible set is nonempty. The cutoff r^* is increasing in μ_H and decreasing in c, and it satisfies $\lim_{|\mu_H - c| \to 0} r^* = 0$ and $\lim_{\mu_H \to \infty} r^* = \lim_{c \to 0} = \infty$.*

It is easy to verify that the threshold (8) is homogeneous of degree two in (μ_H, μ_L, c), and thus the requirement on agents' patience relaxes when these parameters increase by a common multiplicative factor. This fact leads to a model prediction that struggling organizations could benefit from assigning agents tasks of greater importance, for which effort is more difficult but also easier to monitor as the difference between success and failure is larger.

Conversely, we show that an organization *cannot* be sustained if agents are sufficiently impatient. The intuition for this is straightforward. Each agent is motivated to exert high effort by the threat of eventual removal and by the promise of eventual *vacation*. As he becomes very impatient, the prospect of future sticks and carrots lose their salience, and it becomes impossible to incent high effort. Put another way, when agents become extremely impatient, the principal must reward them with more frequent vacations, but this reduces the overall output of the organization, and eventually there is no positive wage that the principal can promise based on the organization's output.

[14] Given Proposition 2, whenever we state results which condition on the feasible set being nonempty, it should be understood that this condition applies for a nonempty set of input parameter values.

The following proposition gives a sufficient condition for nonexistence in closed form. For a fixed discount factor, nonexistence obtains when the cost of effort is sufficiently large or the value of high effort is sufficiently small.[15]

Proposition 3. *If* $r > \frac{2\mu_H(\mu_H - c)}{\lambda^2}$*, there is no feasible organization.*

Note, however, that while the organization must induce a positive flow wage u to prevent all agents from leaving, that wage can be less than c — agents may receive a negative net flow payoff while in the working state, yet prefer to remain in the organization to obtain a positive net flow payoff while enjoying respite. In other words, agents might be "working for the weekend" in the sense that the only instances at which they receive more flow utility being in the organization than out of it are those when they are not working.

Proposition 4. *For a nonempty set of parameter values, there exists a feasible organization* (u, w^0, w^*) *with* $u < c$.

B. Characterizing the Feasible Set

Having established that the set of feasible organizations can be nonempty, we turn to characterizing this set. As an intermediate step, it is useful to consider a fixed promised utility u and characterize the set of (w^0, w^*) pairs such that (u, w^0, w^*) satisfies (6) and $U(Q(u, w^0, w^*)) \geq u$; in other words, it is the set of organizations whose output is at least u when agents receive flow payoff u. We call this the *u-supportive set* and denote it by S^u.[16]

Consider the effect of marginally increasing w^0 for fixed values of w^* and u subject to (6); the principal implements such a change through later expulsion of agents in a stochastic sense. Since agents enter the organization with higher continuation values, a greater mass of agents in steady state reside at continuation values above w^0, which means there are both more agents in the interval (w^0, w^*) who are working and more agents at w^* who are shirking; that is, both α and ν are increasing in w^0. While it is immediately clear that the organization becomes larger, these forces affect the *average* output of the organization in opposite directions. We show analytically that the increase in shirking dominates, so that the fraction of agents working unambiguously decreases in w^0. This observation implies that the u-supportive set lies below some curve $\bar{w}^0(w^*)$ in (w^*, w^0)-space, defined for w^* in a subset of $(0, u/r)$.

Next, consider fixing u and w^0 while marginally increasing w^*, a change which is implemented by allowing agents to shirk more. This change has no effect on the steady state distribution of agents below w^*, but it expands the distribution to the right, increasing the mass of agents working. The effect of increasing w^* on the

[15] Recall that we have assumed $\mu_L < 0$, so that low effort is inefficient. Without this assumption, there would always exist an (uninteresting) organization in which all agents shirk.

[16] For visual depictions of the u-supportive set, it is convenient to place w^* on the horizontal axis and w^0 on the vertical axis.

fraction of agents shirking is ambiguous; we show that the latter is quasiconcave in the former. For small w^*, say very close to w^0, increasing w^* can improve the average effort in the organization by delaying the time at which new agents get to shirk. On the other hand, as w^* increases, agents must spend a greater amount of time shirking at w^* — the reflecting barrier at w^* becomes stickier, and this eventually outweighs the increase in the mass of working agents. This quasiconcavity implies that the horizontal cross sections of the u-supportive set are intervals.

Finally, we note that a "wedge" always exists between the bottom left of the u-supportive set and the 45-degree line. In other words, when new agents start at very low continuation values and are highly likely to be expelled in the very near future, the designer must force them to work for some time before earning respite, or else the output of the organization will not exceed the promised wage.

Figure 2 shows the u-supportive set for a fixed value of u.

Proposition 5. *There exist $\underline{u}, \overline{u} \in (0, \mu_H)$ such that if $u < \underline{u}$ or $u > \overline{u}$, the u-supportive set is empty. When it is nonempty, the u-supportive set for any $u \in (0, \mu_H)$ can be written as $\{(w^*, w^0) \in \mathbb{R}^2_+ : w^0 \in (0, \bar{w}^0(w^*)]\}$, where $\bar{w}^0$ is a single-peaked function taking values in $[0, w^*]$. If w^* is such that $\bar{w}^0(w^*) < w^*$, then $\bar{w}^0$ is continuously differentiable at w^*. For sufficiently small $w^0 > 0$, $w^* > w^0$ whenever (w^*, w^0) is in the u-supportive set.*

From the definitions, if an organization (u, w^0, w^*) is feasible, then (w^0, w^*) is u-supportive. Though not immediate from the definition, a converse to this statement is also true, as reported in the following proposition. The value of these facts are that the u-supportive sets fully determine the feasible set, which aids us in solving the principal's problem in the next section.

Proposition 6. *An organization (u, w^0, w^*) is feasible if and only if (w^0, w^*) is $\tilde{u}$-supportive for some $\tilde{u} \in (0, \mu_H)$. Hence, $\mathcal{S} = \{(\tilde{u}, w^0, w^*) : (w^0, w^*) \in S^{\tilde{u}}$ and $\tilde{u} \in (0, \mu_H)\}$. There is a nonempty set of parameter values such that $w^0 < w^*$ for all $(u, w^0, w^*) \in \mathcal{S}$ and $\mathcal{S}$ is nonempty.*

The last part of the proposition says that for some parameter settings, the principal must force agents to wait some time before enjoying respite in order to create a feasible organization; in other words, all the u-supportive sets lie below the 45-degree line. We provide a sufficient condition, in closed form, on parameter values for this to be the case.

Armed with the results of this section, we turn to the principal's problem of choosing among feasible organizations to maximize one of two possible objective functions.

V. The Principal's Problem(s)

We now analyze the problem of a principal who seeks to design an organization to maximize an objective $V(u, w^0, w^*)$ over the set of feasible organizations. We focus

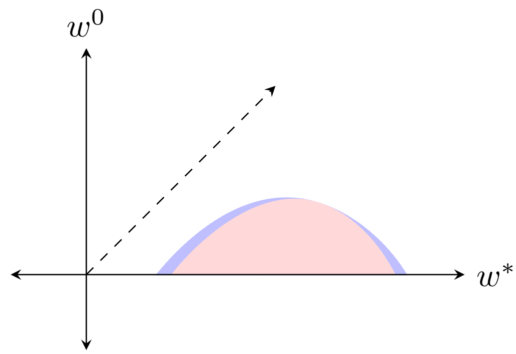

Figure 2. : A projection of the feasible set (light blue) overlayed with a u-supportive set (light red) for $(\mu_H, \mu_L, c, r, \psi) = (.5, -.5, .14, 1, 1)$ and $u = .46$. The dashed line is the 45-degree line.

on two distinct objectives: maximizing the organization's size and maximizing per capita output.

A. *Maximizing Organizational Size*

Suppose the organization designer wishes to maximize the total size of the organization. For example, a social networking website might want to attract as many users as possible in order to maximize advertising revenue; whether those users are engaging in high quality interactions with each other is of lesser importance, conditional on their willingness to keep using the platform. The designer's problem is to maximize $V(u, w^0, w^*) = \alpha(u, w^0, w^*) + \nu(u, w^0, w^*)$ subject to (6) and (7).

By Proposition 6, this problem can be solved in a two-step optimization, with the principal first picking an arbitrary $u \in (0, \mu_H)$ and optimizing the size of the organization within the u-supportive set, and then (working backwards) optimizing over u. As argued in the previous section, both α and ν are increasing in both w^0 and w^* within the u-supportive set for any u. Hence, a solution to the first step must lie on the northeastern frontier of the u-supportive set.

In case parameter values are in the set described by the last statement of Proposition 6, any such solution must lie on the *interior* of this northeastern frontier. To see this, note that the designer can trade off a marginal reduction in w^0 for a (relatively) arbitrarily large increase in w^*. In other words, if the u-supportive set lies below the 45-degree line, the top of this set is flat, and the indifference curve for organizational size which intersects the top of the feasible set must cut into the feasible set, so there exist other points in the u-supportive set which yield a larger organizational size. This implies that if (u, w^0, w^*) is a size-maximizing organization, (w^0, w^*) does not lie at the top of the union of S^u.

Intuitively, the principal finds it worthwhile to limit the lifetime value promised to incoming agents by increasing the performance demands on these agents for earning respite; the resulting increase in the fraction of agents working "buys" the principal the ability to more significantly increase the size of the organization by providing later, but more frequent, respite.

In the second step of the optimization, the principal faces a trade-off, as the u-supportive sets vanish as u becomes sufficiently small or sufficiently large. Hence, any optimal u lies in $[\underline{u}, \overline{u}]$ as defined in Proposition 5.

Proposition 7. *If the feasible set is nonempty, there exists a feasible organization which maximizes total size across all feasible organizations, and it lies on the northeastern frontier of its u-supportive set.*

That the u-supportive sets vanish as u becomes sufficiently small highlights a tension facing the principal between organizational size and feasibility. To illustrate this point, consider a fixed (w^0, w^*) pair and vary the flow payoff u exogenously. The lemma below implies that unambiguously, a principal seeking to maximize the organization's size benefits from reducing the payoff u. Effectively, reducing this payoff puts upward pressure on the drift of agents' continuation values, as expressed in (2). This results in later expulsion of agents and a larger organization overall. However, as the discussion above demonstrates, eventually the average output of the organization deteriorates as the proportion of working agents decreases.

Lemma 1. *For $0 < w^0 \leq w^* < u/r$, f_- and f_+ are decreasing in u, pointwise w.r.t. w, and both α and ν are decreasing in u. Hence, total organizational size is decreasing in u.*

An implication of this result is that if the principal's goal is to maximize the organization's size, then whenever the principal is faced with multiple fixed point values of u for a given (w^0, w^*), the principal should choose to implement the *lowest* of these fixed points. In the next section, we consider an objective function of the principal where the opposite is true.

B. Maximizing Per Capita Output

We now consider a designer whose objective is to maximize steady state output per capita. For example, the designer could be the leader of a research lab who is in continual need of assistants to perform specialized tasks. Resources and credit for publications or discoveries must be shared with each member of the team, and therefore the designer's first order concern is the average effort level, not how many people join the team.

Formally, the designer's problem is

$$\max_{(u,w^0,w^*)\in\mathcal{S}} Q(u,w^0,w^*)\mu_H + (1 - Q(u,w^0,w^*))\mu_L$$

subject to (6) and (7), which is equivalent to maximizing $Q(u, w^0, w^*)$, or simply u, over the set of feasible organizations.

So that the problem is nontrivial, suppose that the feasible set is nonempty, and consider a particular u such that the u-supportive set is nonempty. By Lemma A.2, the per-capita output of the organization is strictly decreasing in w^0, so for any fixed w^*, the principal would like to set w^0 arbitrarily close to 0. Consequently, the organization will have virtually no agents. Since we do not allow $w^0 = 0$, there is an open set problem, which we circumvent by characterizing the supremum of per-capita output over the set of feasible organizations with $w^0 > 0$. We can then identify sequences of organizations for which per-capita output converges to this supremum.

It is useful, then, to consider the limit function $Q^0(u, w^*) := \lim_{w^0 \to 0} Q(u, w^0, w^*)$, which is an upper bound on $Q(u, w^0, w^*)$ for $w^0 > 0$. We show that this function, like $Q(u, w^0, w^*)$ for $w^0 > 0$, is single-peaked in w^*, and we denote its unique maximizer by $w^*_{PC}(u)$. Hence, for fixed u, the sequence of average output associated with any sequence of u-supportive (w^0, w^*) converging to $(0, w^*_{PC}(u))$ converges to the supremum over S^u of $Q(u, w^0, w^*)$. It is worth emphasizing that despite the fact that $w^0 \to 0$, $Q(u, w^0, w^*)$ remains bounded away from 1; although new agents become arbitrarily unlikely to reach the shirking state before being removed from the organization, a positive *fraction* of agents remain shirking.

Next, we identify the optimal u. Since the objective is to maximize the fixed point u itself, it is enough to find the supremum, denoted u_{PC}, of the set of u such that $U(Q^0(u, w^*_{PC}(u))) > u$. Although u_{PC} itself is not attainable in a feasible organization — this would require $w^0 = 0$ which is not permitted — u_{PC} can be approximated arbitrarily closely by a sequence of feasible organizations. These results are summarized in the following proposition.

Proposition 8. *If the feasible set is nonempty, then there exists a unique triple* $(u_{PC}, w^0_{PC}, w^*_{PC}) = (u_{PC}, 0, w^*_{PC})$ *with the following properties: (i)* u_{PC} *is the supremum of the per capita output across all feasible organizations, and (ii) the designer can approximate* u_{PC} *arbitrarily closely by choosing feasible organizations arbitrarily close to* $(u_{PC}, w^0_{PC}, w^*_{PC})$*. However, approximating* u_{PC} *requires organizations to be made arbitrarily small.*

It is noteworthy that there is a unique (limiting) organization design which maximizes per capita output, but the principal must heavily sacrifice the organization's size to attempt to implement it. The reason is that the organization most effectively limits the fraction of agents shirking by starting agents at extremely low continuation values; this ensures that they are very unlikely to earn the opportunity to shirk before being expelled from the organization. This result is reminiscent of partnership organizations in which senior (i.e., vested) partners recruit junior colleagues on the lowest rung of the ladder and promote virtually none of them. Interestingly we find that such an organization is itself *very* small in steady state, as partners trade off the size of the organization in order to maintain a high percentage of hard-working juniors.

VI. Discussion

We have proposed a model of organization design in which there is a large number of small agents, whose efforts exert positive externalities but whose interactions are such that high effort can only be incentivized through a central reputation system. Absent transfer payments, agents must be permitted to shirk in some instances after good performance. Under the Brownian monitoring (or output) technology, this implies that agents' continuation values follow a sticky Brownian motion. Using techniques in stochastic calculus, we characterize the steady state of the organization as a stationary distribution over continuation values. Finally, we frame the social planner's optimization problem in terms of the steady state distribution, where she optimizes the design parameters subject to a feasibility constraint. We identify a fundamental trade-off between the size and feasibility of the organization, mediated by the mass of shirking agents.

Our model could also be extended to capture other aspects of real world organizations. For example, we have assumed an exogenous inflow rate of new agents, but one could endogenize the inflow rate, say, to be an increasing function of the starting payoff w^0. We have also assumed that no agents voluntarily leave the organization, but there are several ways that agents could leave an organization in practice. Agents might have idiosyncratic shocks that force them to seperate from the organization, independent of their continuation values; we conjecture that this would simply increase the effective discount factor of agents, and would reduce the size of the feasible set. A more substantively different possibility would be to give agents a positive outside option; this would put a positive lower bound on continuation values and would also restrict the feasible set.

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Appendix

A1. Proofs for Section III

In this section we derive the stationary distribution. In Lemma A.1, we show that the densities must satisfy a standard Kolmogorov forward equation along with a set of boundary conditions. While stationary distributions and their boundary conditions have been treated in numerous texts (for example, see Karlin and Taylor (1981), Gardiner (2009)), we are unaware of an existing result that is sufficiently general to take "off the shelf" by accommodating the sticky reflection, resetting barrier and form of drift for our continuation value process, so we provide a full derivation here which adapts the approach of Harrison and Lemoine (1981).

For future reference, we state the Kolmogorov forward (or Fokker-Planck) equation

$$rf(w) + (rw - (u - c))f'(w) = \frac{\lambda^2}{2}f''(w) \tag{A1}$$

and define the general form of its solution up to two arbitrary constants as follows:

$$f(w) = e^{\gamma(w)^2}\left(C_1 + C_2 \text{erf}\left\{\gamma(w)\right\}\right), \tag{A2}$$

where the functions γ and erf were defined in Section III.

Lemma A.1. *The steady state distribution of agents is characterized by an atom $\nu\{w^*\}$ and piecewise densities f_- and f_+ of the form (A2) defined on $(0, w^0]$ and $[w^0, w^*)$, respectively, subject to the following boundary conditions:*

1) $f_-(0) = 0$.
2) $f_-(w^0) = f_+(w^0)$.
3) $f'_-(0+) = f'_-(w^0-) - f'_+(w^0+)$.
4) $\frac{\lambda^2}{2} f_+(w^*-) + \rho(w^*)\nu\{w^*\} = 0$.
5) $\frac{\lambda^2}{2} f'_-(0+) = \psi$.

Proof of Lemma A.1. We follow the approach in Harrison and Lemoine (1981, pp. 220-221) who derive the stationary distribution for a sticky Brownian motion with constant (negative) drift on $[0, \infty)$ and sticky reflection at 0; the main modifications are to account for the resetting barrier and state-dependent drift. The infinitesimal generator of the W process is the operator Γ defined by

$$\Gamma h(w) = \lim_{dt \downarrow 0} \frac{\mathbb{E}_w[h(W_{dt})] - h(w)}{dt}.$$

For all $w > 0$ and functions h in a suitable domain, the above limit is well-defined and takes values

$$\Gamma h(w) = \begin{cases} (rw^* - u)h'(w^*) & \text{if } w = w^* \\ (rw - (u-c))h'(w) + \frac{\lambda^2}{2} h''(w) & \text{if } w \in (0, w^*). \end{cases}$$

In particular, the above is valid for all h such that h is twice continuously differentiable and bounded and that $\Gamma h(w)$ is continuous, including at w^*. For $w = 0$, the generator is not defined since the jump from 0 to w^0 is instantaneous. For convenience, define $\mu(w) := rw - (u - c)$ and recall that $\rho(w) := rw - u$.

Now a measure ν is a stationary distribution on $[0, w^*]$ for W if and only if for all $t \geq 0$ and all admissible h, we have

$$\int_{[0,w^*]} h(w)\nu(dw) = \int_{[0,w^*]} \mathbb{E}_w[h(W_t)]\nu(dw). \tag{A3}$$

Essentially, this condition says that any statistic of a stationary distribution is unchanging over time. In order to characterize a steady state distribution, we want to transform the right side of the above expression into terms involving only $h''(w)$ and $h'(w^*)$. Now we must have and $\nu\{0\} = 0$ since otherwise the outflow of agents would be of higher order than the inflow. Hence, expanding the right hand side using Ito's formula, restricting the integral to $w \in (0, w^*]$ so that the generator is defined, we have

$$\int_{[0,w^*]} \mathbb{E}_w[h(W_t)]\nu(dw) = \int_{(0,w^*]} \left(h(w) + \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)ds + \sum_{0<s\leq t} \Delta h(W)_s \right] \right) \nu(dw)$$

where $\Delta h(W)_s := h(W_s) - h(W_{s-})$ for $s > 0$. Subtracting the left hand side of (A3) from this, we have

$$\begin{aligned} 0 &= \int_{(0,w^*]} \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)\, ds + \sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \\ &= \int_{(0,w^*]} \mathbb{E}_w \left[\int_0^t \Gamma h(W_s)\, ds \right] \nu(dw) + \int_{(0,w^*]} \mathbb{E}_w \left[\sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw). \end{aligned}$$

Dividing through by t and taking limits as $t \to 0$ yields

$$\begin{aligned} 0 &= \int_{(0,w^*]} \Gamma h(w)\nu(dw) + \lim_{t\to 0} \frac{1}{t} \int_{[0,w^*]} \mathbb{E}_w \left[\sum_{0<s\leq t} \Delta h(W)_s \right] \nu(dw) \\ &= \int_{(0,w^*]} \Gamma h(w)\nu(dw) + \frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)). \end{aligned} \tag{A4}$$

To obtain the second term of (A4), note that $\Delta h(W)_s > 0$ only when $W_{s-} = \lim_{t\to s} W_t = 0$, and in these cases we have $\Delta h(W)_s = h(W_s) - h(W_{s-}) = h(w^0) - h(0)$. To a first order approximation, the second term of (A4) is thus the expectation, over starting points w, of the size of a single jump, $h(w^0) - h(0)$, times the probability that the process starting from w reaches 0 (the probability of 2 or more jumps may be ignored since once the process resets at w^0 it is very far away from 0). After taking the limit, the second term above is $h(w^0) - h(0)$ times the flow rate of mass hitting 0, which is $\lambda^2 f'_-(0+)/2$.

The first term of (A4) can be expanded as

$$\begin{aligned}(\text{A5}) \quad &\int_{(0,w^*)} \left[\mu(w)h'(w) + \frac{\lambda^2}{2}h''(w)\right]\nu(dw) + \rho(w^*)h'(w^*)\nu\{w^*\} \\ &= \int_{(0,w^*)} \mu(w)h'(w)\nu(dw) + \int_{(0,w^*)} \frac{\lambda^2}{2}h''(w)\nu(dw) + \rho(w^*)h'(w^*)\nu\{w^*\}.\end{aligned}$$

We now focus on the first term of (A5). The integral can be split into two regions, $(0, w^0)$ and $[w^0, w^*)$, where the stationary distribution has a density $f_\pm$ of the form (A2). Then, by writing $h'(w) = h'(w^0) - \int_w^{w^0} h''(y)dy = h'(w^0) + \int_{w^0}^{w} h''(y)dy$, the first term of (A5) is equivalent to

$$\begin{aligned}&\int_{(0,w^0)} \mu(w)\left[h'(w^0) - \int_{(w,w^0)} h''(y)dy\right] f_-(w)dw \\ &\qquad + \int_{[w^0,w^*)} \mu(w)\left[h'(w^0) + \int_{(w^0,w)} h''(y)dy\right] f_+(w)dw \\ &= h'(w^0)\int_{(0,w^*)} \mu(w)\nu(dw) - \int_{(0,w^0)} \left[\int_{(0,w)} \mu(y)f_-(y)dy\right] h''(w)dw \\ &\qquad + \int_{[w^0,w^*)} \left[\int_{[w,w^*)} \mu(y)f_+(y)dy\right] h''(w)dw.\end{aligned}$$

Substituting the above into (A5), the first term of (A4) becomes

$$\begin{aligned}\int_{(0,w^*]} \Gamma h(w)\nu(dw) &= \int_{(0,w^0)} h''(w)\left[\frac{\lambda^2}{2}f_-(w) - \int_{(0,w)} \mu(y)f_-(y)dy\right] dw \\ &\quad + \int_{[w^0,w^*)} h''(w)\left[\frac{\lambda^2}{2}f_+(w) + \int_{[w,w^*)} \mu(y)f_+(y)dy\right] dw \\ (\text{A6}) \qquad &\quad + h'(w^0)\int_{(0,w^*)} \mu(w)\nu(dw) \\ &\quad + \rho(w^*)h'(w^*)\nu\{w^*\}.\end{aligned}$$

The first two terms of (A6), having integrals involving $h''(w)$ as coefficients, are all set. Take the last two terms of (A6) and add back in the second term on the RHS of (A4) to write the RHS of (A4) as the sum of the first two terms of (A6) and

$$\frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)) + h'(w^0) \int_{(0,w^*)} \mu(w)\nu(dw) + \rho(w^*)h'(w^*)\nu\{w^*\}. \tag{A7}$$

As noted, the goal is to transform the h involvement above into h'' and $h'(w^*)$ terms. For the first term of (A7), integrate the derivatives twice and exchange the order of integration to get

$$\begin{aligned}
\frac{\lambda^2}{2} f'_-(0)(h(w^0) - h(0)) &= \frac{\lambda^2}{2} f'_-(0) \int_0^{w^0} h'(w)dw \\
&= \frac{\lambda^2}{2} f'_-(0) \int_0^{w^0} \left[h'(w^*) - \int_w^{w^*} h''(y)dy \right] dw \\
&= \frac{\lambda^2}{2} f'_-(0) \left(w^0 h'(w^*) - \int_0^{w^0} \left[\int_w^{w^*} h''(y)dy \right] dw \right) \\
&= \frac{\lambda^2}{2} f'_-(0) \left(w^0 h'(w^*) - \int_0^{w^0} w h''(w)dw - \int_{w^0}^{w^*} w_0 h''(w)dw \right).
\end{aligned} \tag{A8}$$

For the second term of (A7), we have

$$\begin{aligned}
h'(w^0) \int_{(0,w^*)} \mu(w)\nu(dw) &= \left(h'(w^*) - \int_{w^0}^{w^*} h''(y)dy \right) \int_{(0,w^*)} \mu(w)\nu(dw) \\
&= h'(w^*) \int_{(0,w^*)} \mu(w)\nu(dw) - \int_{w^0}^{w^*} h''(w) \left[\int_{(0,w^*)} \mu(y)\nu(dy) \right] dw
\end{aligned} \tag{A9}$$

where we have swapped w and y as variables of integration for later convenience. Plugging (A8) and (A9) back into (A7) and adding back in the first two terms of (A6), we can write (A4) as

$$0 = h'(w^*)M^* + \int_{(0,w^0)} h''(w)M_-(w)dw + \int_{[w^0,w^*)} h''(w)M_+(w)dw, \tag{A10}$$

where we define

$$M^* := \frac{\lambda^2}{2} f'_-(0)w^0 + \int_{(0,w^*)} \mu(w)\nu(dw) + \rho(w^*)\nu\{w^*\},$$
$$M_-(w) := -\frac{\lambda^2}{2} f'_-(0)w + \frac{\lambda^2}{2} f_-(w) - \int_0^w \mu(y) f_-(y)dy,$$
$$M_+(w) := -\frac{\lambda^2}{2} f'_-(0)w^0 + \frac{\lambda^2}{2} f_+(w) + \int_w^{w^*} \mu(y) f_+(y)dy - \int_{(0,w^*)} \mu(y)\nu(dy).$$

Equation (A10) is exactly what we are after. It allows us to completely characterize the steady state distribution. Specifically, because $h'(w^*)$ and $h''(w)$ are completely free (up to the differentiability conditions), the expressions attached to them must all vanish:

$$M^* = 0$$
$$M_-(w) \equiv 0$$
$$M_+(w) \equiv 0.$$

Equation (A10) has several implications. First, from $M''_-(w) = 0$ and $M''_+(w) = 0$, we recover the Kolmogorov forward equation (A1) on the left and right pieces. In addition,

1) $M_-(0+) = 0$ implies $f_-(0+) = 0$
2) $M_-(w^0) = M_+(w^0)$ implies $f_-(w^0) = f_+(w^0)$
3) $M'_-(w^0-) = M'_+(w^0+)$ implies $f'_-(0+) = f'_-(w^0-) - f'_+(w^0+)$
4) $M^* + M_+(w^*) = 0$ implies $\frac{\lambda^2}{2} f_+(w^*-) + \rho(w^*)\nu\{w^*\} = 0$.

Finally, since the outflow of agents must equal the inflow, we have the condition $\frac{\lambda^2}{2} f'_-(0+) = \psi$ which pins down the scale of the distribution. □

Proof of Proposition 1. By Lemma A.1, the steady state distribution of agents can be described by the densities $f_\pm(w) = e^{\gamma(w)^2}(C_1^\pm + C_2^\pm \text{erf}\{\gamma(w)\})$ and an atom $\nu\{w^*\}$ subject to the stated constraints.

As $f_-(0) = 0$, we have:

$$0 = e^{\frac{(u-c)^2}{\lambda^2 r}} \left(C_1^- + C_2^- \text{erf} \left\{ \frac{-(u-c)}{\lambda\sqrt{r}} \right\} \right)$$

Because the error function has odd symmetry, this means that

$$C_1^- - C_2^- \text{erf} \left\{ \frac{u-c}{\lambda\sqrt{r}} \right\} = 0$$

Knowing too that $f'_-(0) = \frac{2\psi}{\lambda^2}$, and by differentiating $f_-(w)$, we get:

$$\begin{aligned}\frac{2\psi}{\lambda^2} &= 2\gamma(0)\gamma'(0)e^{\gamma(0)^2}\left(C_2^-\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + C_2^-\text{erf}\left\{\gamma(0)\right\}\right) \\ &\quad + e^{\gamma(0)^2}C_2^-\gamma'(0)\text{erf}'\{\gamma(0)\} \\ &= 2e^{\gamma(0)^2}C_2^-\gamma'(0)\frac{e^{-\gamma(0)^2}}{\sqrt{\pi}} = \frac{2C_2^-\sqrt{r}}{\lambda\sqrt{\pi}} \\ \implies C_2^- &= \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}.\end{aligned}$$

Thus the lower segment of the distribution function is

$$f_-(w) = e^{\gamma(w)^2}\left(\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w)\right\}\right]\right). \tag{A11}$$

Since $f_-(w)$ and $f_+(w)$ must agree at w^0, we set the lower and upper f functions equal at w^0 to get

$$\begin{aligned}f_+(w^0) = f_-(w^0) &= e^{\gamma(w^0)^2}\left(\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right]\right) \\ &= e^{\gamma(w^0)^2}(C_1^+ + C_2^+\text{erf}\left\{\gamma(w^0)\right\})\end{aligned}$$

and thus, by rearranging terms, we find that

$$C_1^+ = \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right] - C_2^+\text{erf}\left\{\gamma(w^0)\right\}.$$

Therefore,

$$f_+(w) = \\ e^{\gamma(w)^2}\left(\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right] - C_2^+\text{erf}\left\{\gamma(w^0)\right\} + C_2^+\text{erf}\left\{\gamma(w)\right\}\right)$$

and, differentiating both $f_+(w)$ and $f_-(w)$, we get

$$f'_+(w) = 2\gamma(w)\gamma'(w)f_+(w) + e^{\gamma(w)^2}C_2^+\text{erf}'\{\gamma(w)\}\gamma'(w)$$

and

$$f'_-(w) = 2\gamma(w)\gamma'(w)f_-(w) + \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}e^{\gamma(w)^2}\text{erf}'\{\gamma(w)\}\gamma'(w).$$

Because $f'_-(0+) = f'_-(w^0-) - f'_+(w^0+)$, it must be that

$$C_2^+ = \frac{\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[e^{\gamma(w^0)}\text{erf}'\{\gamma(w^0)\}\gamma'(w^0) - e^{\gamma(0)^2}\text{erf}'\{\gamma(0)\}\gamma'(0)\right]}{e^{\gamma(w^0)^2}\text{erf}'\{\gamma(w^0)\}\gamma'(w^0)} = 0$$

Thus, the upper segment of the distribution function is

$$f_+(w) = e^{\gamma(w)^2}\frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}\left[\text{erf}\left\{\frac{u-c}{\lambda\sqrt{r}}\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right] \tag{A12}$$

Finally, completing the derivation of the distribution of agents in the steady state, the mass of agents at w^*, $\nu\{w^*\}$, satisfies

$$\nu\{w^*\} = \frac{\lambda^2 f_+(w^*)}{2(u - rw^*)}.$$

□

A2. Proofs for Section IV

Lemma A.2. *For all fixed u, w^* such that $0 < w^* < u/r$ and all $w^0 \in (0, w^*]$, both $\alpha(u, w^0, w^*)$ and $\nu(u, w^0, w^*)$ are increasing in w^0 and $Q(u, w^0, w^*)$ is strictly decreasing in w^0.*

Proof. We have $Q = \frac{\alpha}{\alpha+\nu} = \frac{1}{1+\frac{\nu}{\alpha}}$ which is decreasing iff $\frac{\nu}{\alpha}$ is increasing, which is true iff $\frac{\nu_{w^0}}{\nu} > \frac{\alpha_{w^0}}{\alpha}$. Using $X := \frac{u-c}{\lambda\sqrt{r}}$ and $Y := \frac{\psi\sqrt{\pi}}{\lambda\sqrt{r}}$ and expanding, these quantities are

$$\begin{aligned}
\nu =& \frac{\lambda^2}{2(u-rw^*)}f_+(w^*) = \frac{\lambda^2}{2(u-rw^*)}e^{\gamma(w^*)^2}Y\left(\text{erf}\left\{X\right\} + \text{erf}\left\{\gamma(w_0)\right\}\right) \\
\nu_{w^0} =& \frac{\psi}{u-rw^*}e^{\gamma(w^*)^2-\gamma(w^0)^2} > 0 \\
\alpha =& \int_0^{w^0} f_-(w)\,dw + \int_{w^0}^{w^*} f_+(w)\,dw \\
=& \int_0^{w^0} e^{\gamma(w)^2}Y\left[\text{erf}\left\{X\right\} + \text{erf}\left\{\gamma(w)\right\}\right]\,dw \\
&+ \int_{w^0}^{w^*} e^{\gamma(w)^2}Y\left[\text{erf}\left\{X\right\} + \text{erf}\left\{\gamma(w^0)\right\}\right]\,dw
\end{aligned}$$

$$\alpha_{w^0} = \int_{w^0}^{w^*} e^{\gamma(w)^2} Y \frac{2e^{-\gamma(w^0)^2}}{\sqrt{\pi}} \frac{\sqrt{r}}{\lambda}\, dw$$
$$= \frac{2\sqrt{r}e^{-\gamma(w^0)^2}}{\lambda\sqrt{\pi}\left[\mathrm{erf}\left\{X\right\} + \mathrm{erf}\left\{\gamma(w^0)\right\}\right]} \int_{w^0}^{w^*} f_+(w)\, dw > 0.$$

Define $Z := \frac{2\sqrt{r}e^{-\gamma(w^0)^2}}{\lambda\sqrt{\pi}[\mathrm{erf}\{X\}+\mathrm{erf}\{\gamma(w^0)\}]} > 0$ to be the constant outside the integral of the last expression. By canceling terms, we have $\frac{\nu_{w^0}}{\nu} = Z$, whereas

$$\text{(A13)} \qquad \frac{\alpha_{w^0}}{\alpha} = Z \frac{\int_{w^0}^{w^*} f_+(w)\, dw}{\int_0^{w^0} f_-(w)\, dw + \int_{w^0}^{w^*} f_+(w)\, dw} < Z,$$

so we are done. □

The next lemma implies that for any fixed $w^0 > 0$, there is a single (possibly empty) interval of w^* values for which the organization is feasible.

Lemma A.3. *For all fixed u, w^0 such that $0 < w^0 < u/r$ and for all $w^* \in [w^0, u/r)$, $\alpha(u, w^0, w^*)$ is increasing in w^* and Q is quasiconcave in w^*. In particular, Q is eventually decreasing, and if $Q_{w^*} = 0$ for some value of w^*, then Q_{w^*} is decreasing in w^* at that point.*

Proof. To abbreviate, we use prime notation to denote the derivatives with respect to w^*, and as f_+ and its derivatives are to be evaluated at w^*, we suppress dependence on w^*. Immediately, we have $\alpha' = f_+ > 0$, and from $\nu = \frac{\lambda^2}{2}\frac{f_+}{u-rw^*}$ we have $\nu' = \frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+'+rf_+}{(u-rw^*)^2}\right]$. Now $Q = \frac{\alpha}{\alpha+\nu} = g\left(\frac{\nu}{\alpha}\right)$ where $g(x) := \frac{1}{1+x}$. Taking derivatives, we have $Q' = g'\left(\frac{\nu}{\alpha}\right)\left(\frac{\nu}{\alpha}\right)'$. Now Q is decreasing if and only if $\left(\frac{\nu}{\alpha}\right)' > 0$ which is equivalent to $\alpha\nu' - \nu\alpha' > 0$. Expanding yields

$$\alpha\nu' - \nu\alpha' = \alpha\frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+' + rf_+}{(u-rw^*)^2}\right] - \frac{\lambda^2}{2}\frac{f_+^2}{u-rw^*}$$
$$= \frac{\lambda^2}{2(u-rw^*)}\left\{\alpha\left[f_+' + \frac{rf_+}{(u-rw^*)}\right] - f_+^2\right\}.$$

Inside the braces, as $w^* \to u/r$, α and f_+ have positive, finite limits and f_+' has a finite limit, while $\alpha\frac{rf_+}{u-rw^*} \to +\infty$. Hence the expression tends to $+\infty$ as $w^* \to u/r$, and in particular, it is positive for sufficiently large w^*. We conclude that Q is eventually decreasing.

Next, suppose $Q' = 0$, which is equivalent to $\alpha\nu' - \nu\alpha' = 0$. As noted above, $\alpha' > 0$ and hence if $Q' = 0$ we have $\nu' > 0$. Further differentiation yields $Q'' = g''\left(\frac{\nu}{\alpha}\right)\left[\left(\frac{\nu}{\alpha}\right)'\right]^2 + g'\left(\frac{\nu}{\alpha}\right)\left(\frac{\nu}{\alpha}\right)''$. Using $Q' = 0$ and $g' < 0$, we have $Q'' < 0$ if and only if

$$\left(\frac{\nu}{\alpha}\right)'' > 0 \iff \alpha^2\nu'' - \alpha\nu\alpha'' > 2\alpha\alpha'\nu' - 2\nu(\alpha')^2 \iff \nu''\alpha' > \alpha''\nu'.$$

where we have used that $\alpha = \alpha'\frac{\nu}{\nu'}$ to obtain the last inequality. We now establish this last inequality. Differentiate v' to obtain

$$\begin{aligned}
\nu'' &= \frac{\lambda^2}{2}\left[\frac{(u-rw^*)^3 f_+'' + 2r(u-rw^*)^2 f_+' + 2r^2(u-rw^*)f_+}{(u-rw^*)^4}\right] \\
&> \frac{\lambda^2}{2}\left[\frac{(u-rw^*)^3 f_+'' + r(u-rw^*)^2 f_+' + r^2(u-rw^*)f_+}{(u-rw^*)^4}\right] \\
&> \frac{\lambda^2}{2}\left[\frac{(u-rw^*) f_+'' + r f_+'}{(u-rw^*)^2}\right],
\end{aligned}$$

where in the first inequality we have used from $v' > 0$ that $(u-rw^*)f_+' + rf_+ > 0$ and in the second inequality we have used $(u-rw^*)f_+ > 0$. Recall that $\alpha' = f_+$ and $\alpha'' = f_+'$, and thus a sufficient condition for our desired inequality $\nu''\alpha' > \alpha''\nu'$ is

$$\frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+'' + rf_+'}{(u-rw^*)^2}\right] f_+ > f_+'\frac{\lambda^2}{2}\left[\frac{(u-rw^*)f_+' + rf_+}{(u-rw^*)^2}\right] \iff f_+''f_+ > (f_+')^2.$$

Using $f_+(w^*) = e^{\gamma(w^*)^2} Y[\text{erf}\{X\} + \text{erf}\{\gamma(w^0)\}]$ and (by definition) $\gamma(w) = \frac{rw-(u-c)}{\lambda\sqrt{r}}$, this inequality can be written as

$$[(2\gamma\gamma')^2 f_+ + 2(\gamma')^2 f_+]f_+ > (2\gamma\gamma')^2(f_+)^2$$

which clearly holds. We conclude that $Q' = 0$ implies $Q'' < 0$, so Q is quasiconcave in w^*. □

Proof of Proposition 2. For any fixed w^0 and w^* with $0 < w^0 \leq w^*$, $\alpha(u, w^0, w^*)$ and $f_+(w^*)$ (where dependence of the latter on u and w^0 has been suppressed) have finite limits as $u \downarrow rw^*$, while $\nu = \frac{\lambda^2}{2(u-rw^*)} f_+(w^*) \to +\infty$, and thus

$$\begin{aligned}
\lim_{u\downarrow rw^*} Q(u, w^0, w^*) &= \lim_{u\downarrow rw^*} \frac{\alpha(u, w^0, w^*)}{\alpha(u, w^0, w^*) + \nu(u, w^0, w^*)} \\
&= 0.
\end{aligned}$$

In what follows, we show there exists $w^* \in (0, (\mu_H - c)/r)$ such that for sufficiently small w^0, when $u = rw^* + c$, output strictly exceeds u, and hence by the intermediate value theorem, there exists a fixed point u. Take $w^* \in (0, \mu_H - c)/r)$ and

$u = rw^* + c \in (0, \mu_H)$. Then

$$\alpha(u, w^0, w^*)|_{u=rw^*+c}$$
$$= \int_0^{w^0} Y \exp(r(w-w^*)^2/\lambda^2) \left[\operatorname{erf}\left\{\frac{\sqrt{r}w^*}{\lambda}\right\} + \operatorname{erf}\left\{\frac{\sqrt{r}(w-w^*)}{\lambda}\right\}\right] dw$$
$$+ \int_{w^0}^{w^*} Y \exp(r(w-w^*)^2/\lambda^2) \left[\operatorname{erf}\left\{\frac{\sqrt{r}w^*}{\lambda}\right\} + \operatorname{erf}\left\{\frac{\sqrt{r}(w^0-w^*)}{\lambda}\right\}\right] dw,$$
$$\nu(u, w^0, w^*)|_{u=rw^*+c} = \frac{\lambda^2}{2c} Y \left[\operatorname{erf}\left\{\frac{\sqrt{r}w^*}{\lambda}\right\} + \operatorname{erf}\left\{\frac{\sqrt{r}(w^0-w^*)}{\lambda}\right\}\right].$$

Now $\lim_{w^0 \downarrow 0} Q(rw^* + c, w^0, w^*)$ is of the form $\frac{0}{0}$, and by L'Hôpital's rule,

$$\lim_{w^0 \downarrow 0} Q(rw^* + c, w^0, w^*) = \lim_{w^0 \downarrow 0} \frac{\alpha_{w^0}(rw^* + c, w^0, w^*)}{\alpha_{w^0}(rw^* + c, w^0, w^*) + \nu_{w^0}(rw^* + c, w^0, w^*)}, \qquad \text{where}$$
$$\alpha_{w^0}(rw^* + c, w^0, w^*) = \int_{w^0}^{w^*} Y \exp(r(w-w^*)^2/\lambda^2) \frac{2\sqrt{r}}{\lambda\sqrt{\pi}} \exp(-r(w^0-w^*)^2/\lambda^2)$$
$$\nu_{w^0}(rw^* + c, w^0, w^*) = \frac{\lambda^2}{2c} Y \frac{2\sqrt{r}}{\lambda\sqrt{\pi}} \exp(-r(w^0-w^*)^2/\lambda^2).$$

Taking limits and simplifying, we obtain

$$\lim_{w^0 \downarrow 0} Q(rw^* + c, w^0, w^*) = \frac{\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\, dw}{\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\, dw + \frac{\lambda^2}{2c}}$$
$$=: \hat{Q}(w^*).$$

Next, we show that there exists $w^* \in (0, (\mu_H - c)/r)$ such that

$$\hat{Q}(w^*)\mu_H + (1 - \hat{Q}(w^*))\mu_L > u = rw^* + c. \tag{A14}$$

This inequality is equivalent to

$$\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\, dw (\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*) > 0.$$

The integrand above is bounded below by 1, so the left hand side as a whole is bounded below by

$$w^*(\mu_H - c - rw^*) + \frac{\lambda^2}{2c}(\mu_L - c - rw^*), \tag{A15}$$

which can be written as $g(w^*)$ where $g(x) := Ax^2 + Bx + c$, $A := -r$, $B := \mu_H - c - \frac{\lambda^2 r}{2c}$ and $C := \frac{\lambda^2}{2c}(\mu_L - c)$. Note that g is a concave quadratic function with $g(0) = \frac{\lambda^2}{2c}(\mu_L - c) < 0$, and hence if g has any real roots, either both are positive or both are negative. Moreover, both roots are positive if and only if their sum is positive. Now g has real roots with a positive sum if and only if both of the following conditions hold:

$$0 < B^2 - 4AC = \left(\mu_H - c - \frac{\lambda^2 r}{2c}\right)^2 + 4r\frac{\lambda^2}{2c}(\mu_L - c) \qquad \text{and} \tag{A16}$$

$$0 < -\frac{B}{A} \iff 0 < B = \mu_H - c - \frac{\lambda^2 r}{2c}. \tag{A17}$$

Using $\lambda = \frac{c}{\mu_H - \mu_L}$, these inequalities expand to, respectively,

$$0 < \left(\mu_H - c - \frac{cr}{2(\mu_H - \mu_L)^2}\right)^2 + 2\frac{cr}{(\mu_H - \mu_L)^2}(\mu_L - c) \qquad \text{and} \tag{A18}$$

$$0 < \mu_H - c - \frac{cr}{2(\mu_H - \mu_L)^2}. \tag{A19}$$

Collecting r terms, the right side of (A18) is a convex quadratic in r,

$$h(r) := \frac{c^2}{4(\mu_H - \mu_L)^4}r^2 + \left(\frac{-c^2 + 2c\mu_L - c\mu_H}{(\mu_H - \mu_L)^2}\right) r + (\mu_H - c)^2$$

with sign pattern $+, -, +$. It follows that $h(0) > 0$ and $h'(0) < 0$. The inequality (A19) is equivalent to

$$r < \bar{r} := \frac{2(\mu_H - c)(\mu_H - \mu_L)^2}{c}.$$

For $r = \bar{r}$, the first term on the right side of (A18) vanishes while the second term is negative, so $h(\bar{r}) < 0$. It follows that h has two real roots, both positive. Now $h(r)$ is decreasing for all $r \in [0, \bar{r}]$ and in this interval, $h(r) \geq 0$ if and only if $r < r^*$, where r^* is the lower of the two roots of h, given explicitly by (8). Hence, for $r < r^*$, (A18) and (A19) hold, and therefore g has two positive roots.

It is easy to verify that $g'(x)|_{x=(\mu_H - c)/r} < 0$, and since g is concave, the two roots of g must lie in $(0, (\mu_H - c)/r)$, which implies there exists w^* in this interval such that $g(w^*) > 0$. Retracing the earlier steps, (A14) holds for such w^* and hence by continuity there exists $w^0 \in (0, w^*)$ such that $Q(u, w^0, w^*)\mu_H + (1 - Q(u, w^0, w^*))\mu_L > u$ for $u = rw^* + c$. By the intermediate value theorem, there exists $u \in (rw^*, rw^* + c)$ such that average output under (u, w^0, w^*) is exactly u, and a feasible organization exists.

We claim that r^* is increasing in μ_H and decreasing in c. For μ_H it suffices to show that the term $c+\mu_H-2\mu_L-2\sqrt{(c-\mu_L)(\mu_H-\mu_L)}$ is increasing in μ_H. By direct computation, its derivative w.r.t. μ_H is $1-\frac{\sqrt{c-\mu_L}}{\sqrt{\mu_H-\mu_L}}>0$ as $\mu_H>c$. For c, we have

$$\begin{aligned}\frac{\partial}{\partial c}r^* &= \frac{2(\mu_H-\mu_L)^2}{c^2}\left(-\mu_H+2\sqrt{(c-\mu_L)(\mu_H-\mu_L)}+2\mu_L-c\frac{\mu_H-\mu_L}{\sqrt{(c-\mu_L)(\mu_H-\mu_L)}}\right)\\ &< \frac{2(\mu_H-\mu_L)^2}{c^2}\left(-\mu_H+2\sqrt{(c-\mu_L)(\mu_H-\mu_L)}+2\mu_L-c\right)\\ &= \frac{4(\mu_H-\mu_L)^2}{c^2}\left(\sqrt{(c-\mu_L)(\mu_H-\mu_L)}-\frac{(\mu_H-\mu_L)+(c-\mu_L)}{2}\right)\end{aligned}$$

which is negative by applying the Arithmetic Mean–Geometric Mean inequality to the pair of positive numbers $(c-\mu_L,\mu_H-\mu_L)$.

Now r^* is continuous in (μ_H,μ_L,c) and vanishes when $\mu_H=c$, giving the limit result for $|\mu_H-c|\to 0$. As $\mu_H\to\infty$, note that both the first factor and the second factor are positive and tend to infinity as $\mu_H\to\infty$. As $c\to 0$, the first factor tends to $+\infty$ and the second factor tends to $\left(0+\mu_H-2\mu_L-2\sqrt{(0-\mu_L)(\mu_H-\mu_L)}\right)>0$, giving the last two limits in the proposition. □

The following lemma shows the existence of a "wedge" in the graph of the feasible set.

Lemma A.4. *For all $u\in(0,\mu_H)$, for sufficiently small $w^0>0$, $w^*>w^0$ whenever (w^*,w^0) is in the u-supportive set.*

Proof. We show that $\lim_{w^0\to 0}Q(u,w^0,w^0)=0$. We have $Q(u,w^0,w^0)=\frac{\alpha(u,w^0,w^0)}{\alpha(u,w^0,w^0)+\nu(u,w^0,w^0)}$ and we show that $\lim_{w^0\to 0}\frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)}\to 0$. Expanding,

$$\begin{aligned}\lim_{w^0\to 0}\frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)} &= \lim_{w^0\to 0}\frac{\int_0^{w^0}f_-(w)\,dw}{\frac{\lambda^2}{2(u-rw^0)}f_-(w^0)}\\ &= \frac{\lim_{w^0\to 0}f_-(w^0)}{\lim_{w^0\to 0}\frac{d}{dw^0}\left[\frac{\lambda^2}{2(u-rw^0)}f_-(w^0)\right]}.\end{aligned}$$

The numerator has limit $f_-(0)=0$, while the denominator has limit

$$\begin{aligned}&\lim_{w^0\to 0}\frac{Y\lambda^2}{2\sqrt{\pi}(u-rw^0)^2}\Big[e^{\gamma(w^0)^2}\sqrt{\pi}r\left(\mathrm{erf}\left\{X\right\}+\mathrm{erf}\left\{\gamma(w^0)\right\}\right)\\ &\quad+2(u-rw^0)\gamma'(w^0)\left(1+e^{\gamma(w^0)^2}\sqrt{\pi}\left(\mathrm{erf}\left\{X\right\}+\mathrm{erf}\left\{\gamma(w^0)\right\}\right)\gamma(w^0)\right)\Big]=\frac{\psi}{u}>0,\end{aligned}$$

where we have used that $\lim_{w^0\to 0}\left(\text{erf}\left\{X\right\}+\text{erf}\left\{\gamma(w^0)\right\}\right)=\text{erf}\left\{X\right\}+\text{erf}\left\{-X\right\}=0$. It follows that $\lim_{w^0\to 0}\frac{\alpha(u,w^0,w^0)}{\nu(u,w^0,w^0)}=0$, as desired. □

Lemma A.5. *The limit $Q^0(u,w^*):=\lim_{w^0\to 0}Q(u,w^0,w^*)$ is well-defined, and $Q^0(u,w^*)$ is a differentiable, single-peaked function of w^* which is maximized at some $w^*_{PC}(u)\in(0,u/r)$. Moreover, $Q^0(u,w^*_{PC}(u))<1$.*

Proof. As $\lim_{w^0\to 0}Q(u,w^0,w^*)$ is of the form $0/0$, we use L'Hôpital's rule. Using the expressions from the proof of Lemma A.2,

$$\begin{aligned}\lim_{w^0\to 0}Q(u,w^0,w^*)&=\lim_{w^0\to 0}\frac{\alpha_{w^0}(u,w^0,w^*)}{\alpha_{w^0}(u,w^0,w^*)+\nu_{w^0}(u,w^0,w^*)}\\&=\frac{\int_0^{w^*}e^{\gamma(w)^2}dw}{\int_0^{w^*}e^{\gamma(w)^2}dw+\frac{\lambda^2}{2(u-rw^*)}e^{\gamma(w^*)^2}}\\&=:Q^0(u,w^*).\end{aligned}$$

It is clear that $Q^0(u,w^*)$ is twice continuously differentiable. We argue that $Q^0(u,w^*)$ is single-peaked in w^*, i.e., that $\frac{\partial^2Q^0(u,w^*)}{(\partial w^*)^2}<0$ whenever $\frac{\partial Q^0(u,w^*)}{\partial w^*}=0$. Define $\alpha_0(w^*):=\int_0^{w^*}e^{\gamma(w)^2}dw$ and $\nu_0(w^*):=\frac{\lambda^2}{2(u-rw^*)}e^{\gamma(w^*)^2}$, so that $Q^0(u,w^*)=\frac{\alpha_0(w^*)}{\alpha_0(w^*)+\nu_0(w^*)}$. By arguments in the proof of Lemma A.3, it is enough to show that $\frac{\nu_0''}{\nu_0'}>\frac{\alpha_0''}{\alpha_0'}$ whenever $\left(\frac{\nu_0}{\alpha_0}\right)'=0$, i.e., whenever $\alpha_0\nu_0'=\nu_0\alpha_0'$. Define $f_0(w^*):=e^{\gamma(w^*)^2}$. The rest of the proof of single-peakedness is then isomorphic to the proof of Lemma A.3, since $f_0(w^*)$ is a positive constant multiple of $f_+(w^*)$ (since $w^0>0$ in Lemma A.3).

Next, it is straightforward to verify that $Q^0(u,0)=0$ and $\lim_{w^*\to u/r}Q^0(u,w^*)=0$ and that $Q^0(u,w^*)>0$ for all $w^*\in(0,u/r)$, and since $Q^0(u,w^*)$ is single-peaked in w^*, it attains its maximum on $[0,u/r]$ at some unique $w^*_{PC}(u)\in(0,u/r)$. Finally, it is clear from inspection that $Q^0(u,w^*)<1$ for all $w^*\in(0,u/r)$, so in particular, $Q^0(u,w^*_{PC}(u))<1$. □

Proof of Proposition 3. Suppose $r>\frac{2\mu_H(\mu_H-c)}{\lambda^2}$. By Lemma A.2, if a feasible platform exists, then a feasible platform exists for arbitrarily low $w^0>0$. Fixing u and w^0, by Lemma A.3, the set of w^* such that (w^0,w^*) is u-supportive is an interval $[\underline{w},\overline{w}]$. By Lemma A.4, for sufficiently low $w^0>0$, $\underline{w}>w^0$ which implies that $Q(u,w^0,\underline{w})=Q(u,w^0,\overline{w})$, and by Lemma A.3 there exists a unique $w^*\in[\underline{w},\overline{w}]$ such that $\frac{\partial}{\partial w^*}Q(u,w^0,w^*)=0$, which implies $\alpha_{w^*}\nu=\alpha\nu_{w^*}$. At such a w^*, we have

$$\frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} = \frac{\alpha}{\nu} = \frac{\alpha_{w^*}}{\nu_{w^*}} = \frac{f_+(w^*)}{\frac{\lambda^2}{2(u-rw^*)^2}[(u-rw^*)f_+'(w^*)+rf_+(w^*)]}$$
$$= \frac{f_+(w^*)2(u-rw^*)^2}{\lambda^2[(u-rw^*)2\gamma(w^*)\gamma'(w^*)f_+(w^*)+rf_+(w^*)]}$$
$$= \frac{2(u-rw^*)^2}{2(u-rw^*)(rw^*-(u-c))+r\lambda^2}.$$

The platform (u,w^0,w^*) is not feasible if $\frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} < \frac{u-\mu_L}{\mu_H-u}$; we now show this inequality holds. Using the expression above this is equivalent to

$$\text{(A20)} \qquad \frac{2(u-rw^*)^2}{2(u-rw^*)(rw^*-(u-c))+r\lambda^2} < \frac{u-\mu_L}{\mu_H-u}.$$

The denominator on the left hand side of (A20) must be positive, otherwise we would have $\nu=0$ which is impossible with $w^* < u/r \le \mu_H/r < \infty$. It follows that (A20) is equivalent to

$$\text{(A21)} \quad 0 > 2(u-rw^*)^2(\mu_H-u) - [2(u-rw^*)(rw^*-(u-c))+r\lambda^2](u-\mu_L).$$

The right hand side above is a quadratic function of w^* which is convex since it has coefficient $2r^2(\mu_H-\mu_L)>0$ on $(w^*)^2$. Evaluated at $w^*=u/r$, it simplifies to $-r\lambda^2(u-\mu_L)<0$. Evaluated at $w^*=0$, it simplifies to $k(u) := 2u^2[\mu_H - c - \mu_L] + 2uc\mu_L - r\lambda^2(u-\mu_L)$. We now show that $k(u)<0$. Note that k is a convex, quadratic function of u with $k(0)=r\lambda^2\mu_L<0$. Moreover, $k(\mu_H) = -(\mu_H-\mu_L)[r\lambda^2 - 2\mu_H(\mu_H-c)] < 0$ by assumption. Hence by convexity in w^*, (A21) holds for all u, w^* such that $0 < rw^* < u < \mu_H$, completing the proof. □

Proof of Proposition 4. Define $\chi := \frac{2(\mu_H-c)(\mu_H-\mu_L)^2}{e^{(\mu_H-\mu_L)^2}(c-\mu_L)+2(\mu_H-c)(\mu_H-\mu_L)^2} \in (0,1)$. We show that in particular, if $r<\chi$, there exists a feasible platform with $u<c$. As in the proof of Proposition 2, $\lim_{u\downarrow rw^*} Q(u,w^0,w^*)=0$ for all (w^0,w^*) with $0<w^0\le w^*$, and hence for sufficiently small $u>rw^*$, we have $\mu_H Q(u,w^0,w^*) + \mu_L(1-Q(u,w^0,w^*)<u$. Using $u=c$, we first establish the existence of (w^0,w^*) with $0<w^0\le w^*<u/r$ such that output strictly exceeds u. We have

$$\alpha(u,w^0,w^*)|_{u=c} = \int_0^{w^0} Y\exp(rw^2/\lambda^2)\left[\operatorname{erf}\{0\} + \operatorname{erf}\left\{\frac{\sqrt{r}w}{\lambda}\right\}\right]dw$$
$$+ \int_{w^0}^{w^*} Y\exp(rw^2/\lambda^2)\left[\operatorname{erf}\{0\} + \operatorname{erf}\left\{\frac{\sqrt{r}w^0}{\lambda}\right\}\right]dw$$

$$\nu(u, w^0, w^*)|_{u=c} = \frac{\lambda^2}{2(c - rw^*)} Y \exp(r(w^*)^2/\lambda^2) \left[\text{erf}\,\{0\} + \text{erf}\left\{\frac{\sqrt{r}w^0}{\lambda}\right\}\right].$$

Recall that a platform's output exceeds u if and only if $\frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} > \frac{u-\mu_L}{\mu_H-u}$. By the usual use of L'Hôpital's rule, for $u = c$, we have

$$\lim_{w^0 \downarrow 0} \frac{Q(u, w^0, w^*)}{1 - Q(u, w^0, w^*)} = \frac{\int_0^{w^*} \exp(rw^2/\lambda^2) dw}{\frac{\lambda^2}{2(c-rw^*)} \exp(r(w^*)^2/\lambda^2)}$$
$$= \frac{2(c - rw^*)}{\lambda^2} \int_0^{w^*} \exp[r(w^2 - (w^*)^2)/\lambda^2] dw.$$

Evaluating at $w^* = c/\sqrt{r}$ (which is less than u/r since $r < 1$), the last expression becomes

$$\frac{2(c - c\sqrt{r})}{\lambda^2} \int_0^{c/\sqrt{r}} \exp[(rw^2 - c^2)/\lambda^2] dw,$$

which, using that $r < 1$, is bounded below by $\frac{2(c-c\sqrt{r})}{\lambda^2} \frac{c}{\sqrt{r}} e^{-c^2/\lambda^2} = 2(1/\sqrt{r} - 1)(\mu_H - \mu_L)^2 e^{-(\mu_H-\mu_L)^2}$. Now this expression is strictly decreasing in r, and comparing it to $\frac{c-\mu_L}{\mu_H-c}$ and solving for r yields the sufficient condition stated; when this condition holds, by continuity in w^0, we have for sufficiently small w^0 that $(w^0, w^*) = (w^0, c/\sqrt{r})$ supports u with $\mu_H Q(u, w^0, w^*) + \mu_L(1 - Q(u, w^0, w^*) > u$ for $u = c$. From this and the observation at the beginning of the proof about $u \downarrow rw^*$, the intermediate value theorem gives existence of $u < c$ such that $\mu_H Q(u, w^0, w^*) + \mu_L(1 - Q(u, w^0, w^*) = u$, so we conclude there exists a feasible organization exists with $u < c$. □

Proof of Proposition 5. We must only prove the first claim, since the rest of the proposition is a summary of Lemmas A.2, A.3 and A.4. First, observe that in the proof of Proposition 3, no organization with flow payoff u is feasible if (A21) holds for all $w^* \in (0, u/r)$. By the arguments there, the maximum value (over $w^* \in (0, u/r)$) of the RHS tends to a negative limit as $u \to 0$, and hence by continuity, there exists $\underline{u} \in (0, \mu_H)$ such that for all $u \in (0, \underline{u})$, no u-supportive organization exists. To establish $\overline{u}$, recall that if an organization is feasible, then $\frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} \geq \frac{u-\mu_L}{\mu_H-u}$, and since Q is strictly decreasing in w^0, it must be that $\lim_{w^0 \to 0} \frac{Q(u,w^0,w^*)}{1-Q(u,w^0,w^*)} \geq \frac{u-\mu_L}{\mu_H-u}$. Using now familiar calculations, for any $u \in (c, \mu_H)$, the LHS of this inequality simplifies to

$$\begin{aligned}
&\frac{2(u-rw^*)}{\lambda^2}\int_0^{w^*} e^{\gamma(w)^2-\gamma(w^*)^2}dw \\
&= \frac{2(u-rw^*)}{\lambda^2}\int_0^{w^*} \exp\left[\frac{r(w+w^*)-2(u-c)}{\lambda\sqrt{r}}\frac{\sqrt{r}(w-w^*)}{\lambda}\right]dw \\
&\leq \frac{2(u-rw^*)}{\lambda^2}\int_0^{w^*} \exp\left[\frac{2(u-c)(w^*-w)}{\lambda^2}\right]dw \\
&\leq \frac{2u}{\lambda^2}w^*\exp\left[\frac{2(u-c)w^*}{\lambda^2}\right] \\
&\leq \frac{2\mu_H}{\lambda^2}\frac{\mu_H}{r}\exp\left[\frac{2(\mu_H-c)\mu_H}{r\lambda^2}\right],
\end{aligned}$$

which is a finite bound independent of u. But as $u \to \mu_H$, $\frac{u-\mu_L}{\mu_H-u} \to \infty$, so the necessary condition fails. Hence there exists $\overline{u} \in (c, \mu_H)$ such that the u-supportive set is empty for all $u > \overline{u}$. □

Define $r^{**} : [0,1] \to [0,\infty)$ by

$$r^{**}(\beta) := \frac{(\mu_H-c)^2\beta\left\{3(1-\beta)c + \sqrt{3(1-\beta)c[c(1-\beta)(3+2\beta)+2\beta(\beta\mu_H-\mu_L)]}\right\}}{3\lambda^2[\beta(\mu_H-c)+c-\mu_L]}.$$

Note that $r^{**}(\beta) > 0$ for all $\beta \in (0,1)$. The following lemma provides an alternative sufficient condition for existence; it can be optimized by choosing $\beta = \arg\max_{\tilde{\beta}\in[0,1]} r^{**}(\tilde{\beta})$.

Lemma A.6. *Fix any $\beta \in (0,1)$. If $r < r^{**}(\beta)$, then a feasible platform exists.*

Proof. The proof follows the proof of Proposition 2 through the step where we have reduced the existence problem to establishing the inequality

$$\int_0^{w^*} \exp(r(w-w^*)^2/\lambda^2)\,dw(\mu_H-c-rw^*) + \frac{\lambda^2}{2c}(\mu_L-c-rw^*) > 0$$

for some $w^* \in (\mu_H-c)/r$. In deriving $r^{**}(\beta)$, we use a sharper lower bound on the integrand, but use a stronger sufficient condition in a later step. Specifically, we have $\exp(r(w-w^*)^2/\lambda^2) \geq 1 + r(w-w^*)^2/\lambda^2$, and hence it suffices to show

$$\begin{aligned}
0 &< (\mu_H-c-rw^*)\int_0^{w^*}[1+r(w-w^*)^2/\lambda^2]dw + \frac{\lambda^2}{2c}(\mu_L-c-rw^*) \\
&= w^*[1+r(w^*)^2/(3\lambda^2)](\mu_H-c-rw^*) + \frac{\lambda^2}{2c}(\mu_L-c-rw^*).
\end{aligned}$$

Evaluating the expression above at $w^* = \beta(\mu_H - c)/r$ and multiplying through by r^2 yields an inequality involving a quadratic in r:

$$-\frac{\lambda^2[\beta(\mu_H - c) + c - \mu_L]}{2c} r^2 + \beta(1-\beta)(\mu_H - c)^2 r + \frac{\beta^3(1-\beta)(\mu_H - c)^4}{3\lambda^2} > 0.$$

The left hand size is strictly positive at $r = 0$ and is concave. Its unique positive root is $r^{**}(\beta)$, giving the result. $\square$

Define $\hat{r} := \frac{(\mu_H-\mu_L)^2\left[3(\mu_H-\mu_L)-\sqrt{(\mu_H-\mu_L)(\mu_H-9\mu_L)}\right]\left(\mu_H+3\mu_L+\sqrt{(\mu_H-\mu_L)(\mu_H-9\mu_L)}\right)^2}{8c^2\left[(\mu_H-\mu_L)+\sqrt{(\mu_H-\mu_L)(\mu_H-9\mu_L)}\right]}$, which is a positive real number as $\mu_H > \mu_L$.

Lemma A.7. *If $r > \hat{r}$, then any feasible platform involves $w^0 < w^*$. Moreover, there is a nonempty set of values for (μ_H, μ_L, c, r) such that $r > \hat{r}$ holds while the set of feasible platforms is nonempty.*

Proof. We first show that for $r > \hat{r}$, there is no feasible platform with $w^0 = w^*$. Fix any $u \in (0, \mu_H)$, and consider the function $w^0 \mapsto Q(u, w^0, w^0) = \frac{\int_0^{w^0} f_-(w)dw}{\int_0^{w^0} f_-(w)dw + \nu(w^0)}$. From the proof of Lemma A.4, $\lim_{w^0 \to 0} Q(u, w^0, w^0) = 0$. On the other hand, taking $w^0 \uparrow u/r$, $\int_0^{w^0} f_-(w)dw$ has a finite limit while $\nu(w^0) = \frac{\lambda^2 f_-(w^0)}{2(u - rw^0)} \uparrow +\infty$, so $\lim_{w^0 \uparrow u/r} Q(u, w^0, w^0) = 0$ as well. Now $Q(u, w^0, w^0)$ attains its maximum at some $w^0 \in (0, u/r)$ where the first order condition $0 = \frac{d}{dw^0} Q(u, w^0, w^0) \iff \alpha = \frac{\nu \frac{d\alpha}{dw^0}}{\frac{d\nu}{dw^0}}$ reduces to

$$\alpha = \int_0^{w^0} f_-(w)dw = \frac{f_-(w^0)^2(u - rw^0)}{(u - rw^0)f_-'(w^0) + rf_-(w^0)}. \tag{A22}$$

We now show that at any point satisfying this first order condition, $U(Q(u, w^0, w^0)) = \mu_H Q(u, w^0, w^0) + \mu_L(1 - Q(u, w^0, w^0)) < u$; to do this, we prove the equivalent inequality $\frac{\alpha}{\nu} < \frac{u - \mu_L}{\mu_H - u}$. By (A22),

$$\frac{\alpha}{\nu} = \frac{\frac{f_-(w^0)^2(u-rw^0)}{(u-rw^0)f_-'(w^0)+rf(w^0)}}{\frac{\lambda^2 f_-(w^0)}{2(u-rw^0)}} = \frac{2}{\lambda^2}\frac{f_-(w^0)(u - rw^0)^2}{(u - rw^0)f_-'(w^0) + rf_-(w^0)} < \frac{2(u - rw^0)^2}{\lambda^2 r},$$

where we have used that $f_-'(w^0) > 0$. It suffices then to show that $\frac{2(u-rw^0)^2}{\lambda^2 r} < \frac{u-\mu_L}{\mu_H-u}$, or equivalently $r > \frac{2(u-rw^0)^2(\mu_H-u)}{\lambda^2(u-\mu_L)}$. Since $w^0 \in (0, u/r)$, we have $(u - rw^0)^2 < u^2$, and thus it suffices to show that $r > j(u) := \frac{2u^2(\mu_H-u)}{\lambda^2(u-\mu_L)}$. Our claim is that $\hat{r} = \max_{u \in (0,\mu_H)} j(u)$, so that the condition $r > \hat{r}$ as originally stated is sufficient. Now $j'(u) = -\frac{2}{\lambda^2}\frac{u[2u^2-u(\mu_H+3\mu_L)+2\mu_H\mu_L]}{(u-\mu_L)^2}$. It is easy to verify that there is a unique $u \in (0, \mu_H)$ at which $j'(u) = 0$ and where the maximum is attained, namely $u^* := \frac{1}{4}\left[\mu_H + 3\mu_L + \sqrt{(\mu_H + 3\mu_L)^2 - 16\mu_H\mu_L}\right]$, and $j(u^*) = \hat{r}$.

To prove the second claim of the lemma, it suffices to show that for some $\beta \in (0, 1)$ there exists an instance of (μ_H, μ_L, c) such that $\hat{r} < r^{**}(\beta)$ and thus the set of feasible organizations with $r > \hat{r}$ is nonempty by Lemma A.6. For $\beta = 1/2$, $(\mu_H, \mu_L, c) = (1/32, -1, 7/1024)$ is one such instance. □

Proof of Proposition 6. The "only if" direction is trivial, as we can take $\tilde{u} = u$. For the "if" direction, suppose that $(w^0, w^*) \in S^u$ for some $u \in (0, \mu_H)$, with $U(Q(u, w^0, w^*)) \geq u$; if equality holds, we are finished, so suppose there is strict inequality. Now U is continuous and Q is continuous (in particular in u), and hence $\tilde{u} \mapsto U(Q(\tilde{u}, w^0, w^*)) - \tilde{u}$ is continuous. By assumption, $U(Q(\tilde{u}, w^0, w^*)) - \tilde{u} > 0$ for $\tilde{u} = u$, and $\lim_{\tilde{u}\downarrow rw^*} = U(0) - rw^* < 0$, so by the intermediate value theorem there exists $\tilde{u} \in (rw^*, u)$ such that $U(Q(\tilde{u}, w^0, w^*)) = \tilde{u}$. The second statement of the proposition is merely a translation of the first statement into notation, and the third statement is a summary of Lemma A.7. □

A3. Proofs for Section V

Proof of Proposition 7. First consider any fixed u such that S^u is nonempty and consider a sequence of organizations in S^u for which organizational size converges to the supremum of organizational size across organizations in S^u. Note that the set of such u is a compact subset of $[\underline{u}, \overline{u}]$. This sequence cannot involve $w^0 \to 0$, since then organizational size vanishes. Hence, it lies in a compact subset of S^u and must have a subsequence which converges to a point (u, w^0, w^*) in this compact subset; (u, w^0, w^*) then maximizes the organizational size across feasible organizations. Since organizational size is increasing in w^0, this point must satisfy $w^0 = \overline{w}^0(w^*)$, and moreover, it must lie on the northeastern frontier of the u-supportive set, since organizational size is increasing in w^*. We argue that (u, w^0, w^*) satisfies (7) and is thus feasible. Clearly, this must be true if $w^0 < w^*$, otherwise (u, w^0, w^*) would not be on the northeastern frontier (there would exist $\tilde{w}^* > w^*$ such that $(u, w^0, \tilde{w}^*)$ is also u-supportive). Now if the top of the northeastern frontier includes a point on the 45-degree line, that point is the limit of some sequence of points on the northeastern frontier lying below the 45-degree line, and by continuity, it must be that (7) is satisfied at the top. So we conclude that any size-maximizing organization in the u-supportive set is feasible.

By the maximum theorem, the function mapping u to the maximum organizational size over organizations in S^u is continuous. Since the set of u for which S^u is nonempty is compact, this function attains its maximum and hence there exists a size-maximizing feasible organization, $(u_{OS}, w^0_{OS}, w^*_{OS})$. This organization also maximizes organizational size within $S^{u_{OS}}$ and hence it lies on the northeastern frontier of $S^{u_{OS}}$. □

Proof of Lemma 1. Fix $0 < w^0 \leq w^*$ and consider any u_1, u_2 with $u_2 > u_1 > w^* r$. Let prime notation denote derivatives with respect to w, and consider f_-. Recall that the boundary conditions $f_-(0; u) = 0$ and $f_-'(0+; u) = \frac{2\psi}{\lambda^2}$ are independent of u. Subtracting the ODE (A1) for u_2 from the one for u_1, we have $f_-''(0+; u_1) - f_-''(0+; u_2) = \frac{2}{\lambda^2}(u_2 - u_1) f'(0; u_1) > 0$. Hence, for sufficiently small $w > 0$, $f_-(w; u_1) > f_-(w; u_2)$ and $f_-'(w; u_1) > f_-'(w; u_2)$. We claim that the relationship $f_-'(w; u_1) > f_-'(w; u_2)$, and hence $f_-(w; u_1) > f_-(w; u_2)$, extends to all $w > 0$. If not, let w' be the smallest $w > 0$ for which $f_-(w; u_1) = f_-(w; u_2)$, and let w'' be the smallest $w \in (0, w')$ such that $f_-'(w; u_1) = f_-'(w; u_2) > 0$. Again using subtracting the ODEs, these facts imply that $f_-''(w''; u_1) > f_-''(w''; u_2)$. But this implies that for all $w \in (0, w'')$, $f_-'(w; u_1) < f_-'(w; u_2)$, a contradiction. Hence $f_-(w; u_1) > f_-(w; u_2)$ and $f_-'(w; u_1) > f_-'(w; u_2)$ for all $w > 0$. Turning to f_+, note that both of those inequalities hold in particular at w^0, and hence the third boundary condition from Lemma A.1 implies $f_+'(w^0; u_1) > f_+'(w^0; u_2)$. Now $f_+(w^0; u_1) = f_-(w^0; u_1) > f_-(w^0; u_2) = f_+(w^0; u_2)$, and a similar argument to that above shows that $f_+(w; u_1) > f_+(w; u_2)$ and $f_+'(w; u_1) > f_+'(w; u_2)$ for all $w \in (w^0, w^*]$, establishing the claim for f_+. Since $\alpha = \int_0^{w^0} f_-(w)\, dw + \int_{w^0}^{w^*} f_+(w)\, dw$, where we have established that the integrands are pointwise decreasing in u, α is decreasing in u. Finally, $\nu = \frac{\lambda^2}{2(u - rw^*)} f_+(w^*)$; both factors are decreasing in u, and so is ν. □

Proof of Proposition 8. As argued in the main text, there is no solution to the maximization problem since w^0 is restricted to be positive. Since Q is decreasing in w^0 and $w^*_{PC}(u)$ maximizes $Q(u, w^*)$ (see Lemma A.5), we have $V(u, w^0, w^*) \leq Q^0(u, w^*_{PC}(u))$ for all feasible (u, w^0, w^*). Now define u_{PC} to be the supremum of the set of u such that $U(Q^0(u, w^*_{PC}(u))) > u$, which is well-defined and less than $\overline{u}$ as defined in Proposition 5. Define $w^*_{PC} = w^*_{PC}(u_{PC})$ as the unique maximizer of $Q^0(u_{PC}, w^*)$ with respect to w^*.

Let $\tilde{u}$ denote the supremum of the set of u such that (u, w^0, w^*) is feasible for some w^0, w^*. We show that $u_{PC} = \tilde{u}$. First, note that $u_{PC} \geq \tilde{u}$. Indeed, $\tilde{u} \geq u$ for all feasible (u, w^0, w^*), and $Q^0(u, w^*_{PC}(u)) \geq Q^0(u, w^*) > Q(u, w^0, w^*) = u$, so $u < u_{PC}$. Second, note that $u_{PC} \leq \tilde{u}$, since if $u \leq u_{PC}$ is such that $U(Q^0(u, w^*_{PC}(u))) > u$, there exists by continuity an organization (u, w^0, w^*) such that $U(Q(u, w^0, w^*)) > u$ and hence a feasible organization (u', w^0, w^*) with $u' > u$; since $\tilde{u} \geq u'$, and u can be taken arbitrarily close to u_{PC}, we have $u_{PC} \leq \tilde{u}$. We conclude that u_{PC} is the supremum of per capita output over the set of feasible organizations, and hence u_{PC} can be approximated arbitrarily closely using a sequence of feasible organizations.

Finally, since per capita output is decreasing in w^0 and since w^*_{PC} is the unique maximizer of $Q(u_{PC}, w^*)$ wrt w^*, it follows that if a sequence of feasible organizations converges to a vector other than $(u_{PC}, 0, w^*_{PC}(u_{PC}))$, the principal's value converges to a value less than u_{PC}, so any sequence of feasible organizations for which per capita output converges to u_{PC} must converge to $(u_{PC}, 0, w^*_{PC})$. □